Math 218D Problem Session: Week 12

Answer Key

1. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x,y)=1$ using a computer, and describe the shape (for example, for **a)** you should get an ellipse in $\mathbf{R}^2$, not an elliptic paraboloid in $\mathbf{R}^3$).

(2) Find the 2×2 symmetric matrix $S=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x,y)=\begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2+2bxy+cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x,y)=2x^2+3y^2$

b) $q(x,y)=x^2-5y^2$

c) $q(x,y)=y^2$

d) $q(x,y)=-3x^2-2y^2$

e) $q(x,y)=x^2+3xy+y^2$

f) $q(x,y)=2x^2+4xy+y^2$

g) $q(x,y)=x^2-4xy+5y^2$

Our final two quadratic forms are in 3 variables: this means that S is a 3×3 matrix, and $q(x,y,z)=\begin{pmatrix} x & y & z \end{pmatrix} S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

h) $q(x,y,z)=x^2+y^2+z^2+xy+yz+xz$

i) $q(x,y,z)=x^2+y^2+z^2+2xy+2yz+2xz$

Solution.

a) $q(x,y) = 2x^2 + 3y^2$ has $S = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, which is positive-definite, and $q = 1$ is an ellipse.

b) $q(x,y) = x^2 - 5y^2$ has $S = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

c) $q(x,y) = y^2$ has $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two lines.

d) $q(x,y) = -3x^2 - 2y^2$ has $S = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$, is not positive-definite, and $q = 1$ is empty.

e) $q(x,y) = x^2 + 3xy + y^2$ has $S = \begin{pmatrix} 1 & 3/2 \\ 3/2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

f) $q(x,y) = 2x^2 + 4xy + y^2$ has $S = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

g) $q(x,y) = x^2 - 4xy + 5y^2$ has $S = \begin{pmatrix} 1 & -2 \\ -2 & \\ 5 & \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipse.

h) $q(x,y,z) = x^2 + y^2 + z^2 + xy + yz + xz$ has $S = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipsoid.

i) $q(x,y,z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$ has $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two planes.

2. Diagonalizing quadratic forms

Consider the quadratic form

$$q(x,y) = \frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2.$$

a) What is the symmetric matrix S so that $q(x,y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}$?

b) Find the symmetric diagonalization, $S = QDQ^T$, where the matrix Q is orthonormal. The columns of Q are orthonormal eigenvectors v_1 and v_2, with eigenvalues λ_1 and λ_2.

c) The quadratic form for the diagonal matrix D is $\begin{pmatrix} x_0 & y_0 \end{pmatrix} D \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda_1^2 x_0^2 + \lambda_2^2 y_0^2$. Plot $q(x,y) = 1$ and $\lambda_1 x_0^2 + \lambda_2 y_0^2 = 1$. What is the geometric relationship between these shapes?

d) Confirm that $q(x,y) = \lambda_1 x_0^2 + \lambda_2 y_0^2$, where we relate the variables x_0 and y_0 to the variables x and y using $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = Q^T \begin{pmatrix} x \\ y \end{pmatrix}$.

e) Using **d)**, explain why the equation $q(x,y) = 1$ describes an ellipse. How does this relate to the **eigenvalue test** for positive-definite matrices?

f) Using **d)**, explain why the function $q(x,y)$ is always positive (unless $x = y = 0$). How does this relate to the **positive-energy test** for positive-definite matrices?

g) Consider the **constrained optimization** problem: what is the maximum value of the function $q(x,y)$ on the unit circle $x^2 + y^2 = 1$? at what point (x,y) on the unit circle does q achieve the maximum?
Hint: First answer these questions for the quadratic form $\lambda_1 x_0^2 + \lambda_2 y_0^2$.

Solution.

a) $S = \begin{pmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{pmatrix}$

b) $S = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right)^{-1}$

c) The ellipse $q(x,y) = 1$ is a rotated version of the ellipse $4x_0^2 + 1y_0^2 = 1$.

d) $(x_0, y_0) = Q^T(x,y) = ((1/\sqrt{2})x + (1/\sqrt{2})y, (-1/\sqrt{2})x + (1/\sqrt{2})y)$.

e) In terms of equations and not pictures, we can see that $4x_0^2 + 1y_0^2 = 1$ is an ellipse since both 4 and 1 are positive. Since the change of variables

$$(x_0, y_0) = ((1/\sqrt{2})x + (1/\sqrt{2})y, (-1/\sqrt{2})x + (1/\sqrt{2})y))$$

corresponds to a rotation (Q is a rotation matrix!), this means that $q(x,y) = 1$ is a rotated ellipse.

f) The function $q(x,y) = 4((1/\sqrt{2})x + (1/\sqrt{2})y)^2 + ((-1/\sqrt{2})x + (1/\sqrt{2})y)^2$ is non-negative, as it is a sum of squares with positive coefficients. If it were equal to zero, then both $(1/\sqrt{2})x + (1/\sqrt{2})y$ and $(1/\sqrt{2})x + (1/\sqrt{2})y$ would need to equal zero - but this would mean that $x = y = 0$.

g) The maximum value of $q(x,y)$, constrained to $||(x,y)|| = 1$, is the larger eigenvalue, 4, and is achieved at $(x,y) = \pm 1/\sqrt{2}(1,1)$. The minimum value of $q(x,y)$, constrained to $||(x,y)|| = 1$, is the smaller eigenvalue, 1, and is achieved at $(x,y) = \pm 1/\sqrt{2}(-1,1)$.

3. LDL^T **decomposition**

Find the LDL^T decomposition of the following positive-definite symmetric matrices, by:

(1) Computing the $A = LU$ decomposition.

(2) Setting $D =$ the diagonal matrix with same diagonal entries as U.

a) $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

b) $S = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

Solution.

a) $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This has REF (no scaling or swapping) given by

$$U = \begin{pmatrix} 2 & 1 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = DL^T.$$

Therefore $S = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}^T$.

b) $S = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$ has REF $U = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Therefore $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}^T$.

4. Relation to the quadratic formula

For 2×2 symmetric matrices $S = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, there is an easy test for positive-definiteness, the **discriminant test**:

S is positive-definite if and only if both $a > 0$ and $b^2 - 4ac < 0$.

Let's verify this test in two ways, by relating it to other tests.

a) Method one: Relate the discriminant test to the **determinant test**: S is positive-definite if and only if $\det((a)) > 0$ and $\det\left(\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}\right) > 0$.

b) Method two:

(1) Show that the quadratic form $q(x,y) = (x,y)^T S(x,y)$ equals

$$q(x,y) = ax^2 + bxy + cy^2$$

and factors into

$$q(x,y) = a(x - \frac{-b + \sqrt{b^2 - 4ac}}{2}y)(x - \frac{-b - \sqrt{b^2 - 4ac}}{2}y).$$

(2) If $b^2 - 4ac < 0$, explain why $q(x,y) \neq 0$ for all real numbers x, y (not both zero).

This means that either $q(x,y) > 0$ for all $(x,y) \neq (0,0)$ or $q(x,y) < 0$ for all $(x,y) \neq (0,0)$.

(3) If both $a > 0$ and $b^2 - 4ac < 0$, explain why $q(x,y) > 0$ for all real numbers x, y (not both zero).
Hint: If $a > 0$, can you find a point (x,y) where $q(x,y) > 0$?

This show that **if S satisfies the discriminant test, it satisfies the energy test.**

Solution.

a) The first determinant condition is just $a > 0$. The second determinant is just $ac - (1/4)b^2$. This is positive if and only if $b^2 - 4ac < 0$.

b) (1) You can verify this factorization using the quadratic formula (pretend y is a number, and find the two roots of $ax^2 + (by)x + (cy^2)$: $x = \frac{-by \pm \sqrt{b^2y^2 - 4acy^2}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}y)$.

(2) The only way $q(x,y)$ can equal 0 is if $a = 0$ or if $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}y$. But this latter condition is impossible if $b^2 - 4ac < 0$ and $y \neq 0$, since $x/y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is imaginary while x and y are real, a contradiction.

Now, if $y = 0$ the equation $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}y$ would mean that $x = 0$ as well.
In other words, since $b^2-4ac < 0$ means that $\sqrt{b^2-4ac}$ is imaginary, the only *real* solution to the equation $a(x-\frac{-b+\sqrt{b^2-4ac}}{2}y)(x-\frac{-b-\sqrt{b^2-4ac}}{2}y) = 0$ is $(0,0)$.

(3) If both $a \neq 0$ and $b^2 - 4ac < 0$, the previous step implies that either $q(x, y) > 0$ for all $(x, y) \neq (0,0)$ or $q(x, y) < 0$ for all $(x, y) \neq (0,0)$. This is because a change in sign for $q(x, y)$, on the unit circle $x^2+y^2 = 1$, would require $q(x, y)$ to be zero somewhere on the unit circle, which it is not.
Since $a > 0$, this means that $q(1,0) = a > 0$. Since q is positive at one point, it is positive everywhere except $(0,0)$. Therefore the "positive-energy criterion" is true.
In other words, we have shown that **if S satisfies the discriminant test, it satisfies the energy test.**