1. **Matrices with complex eigenvalues**

Consider the matrices

\[ A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \]

a) Compute the eigenvalues of \( A \) and \( B \). Write each eigenvalue in polar coordinates \( z = re^{i\theta} \).

b) Compute the eigenvectors of \( A \) and \( B \).
2. Some quick matrix exponentials

Compute the matrix exponential $e^A$ of:

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$,

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,

(3) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.
3. A differential equation

Consider the system of differential equations

\[ x'(t) = 3x(t) + 2y(t) \]
\[ y'(t) = 4x(t) - 4y(t) \]

a) Write this as a matrix differential equation

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \]

What is the matrix \( A \)?

b) For this matrix \( A \), find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), as well as the eigenvectors \( w_1 \) and \( w_2 \).

c) Every solution is of the form \((x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2\). If you want the solution to have initial value \((x(0), y(0)) = (1, 1)\), which scalars \( a_1 \) and \( a_2 \) should you choose?

d) Plug the solution with initial value \((x(0), y(0)) = (1, 1)\) to the differential equation, and confirm that it is a solution.

e) For the solution you found in c), compute \((x(1), y(1))\).
4. **A complex ODE**

Consider the system of differential equations

\[ x'(t) = x(t) - y(t), \]
\[ y'(t) = x(t) + y(t). \]

a) Write this as a matrix differential equation

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}. \]

b) Compute the eigenvalues \( \lambda_1, \lambda_2 \) and eigenvectors \( v_1, v_2 \) of the matrix \( A \).

c) Compute the real and imaginary parts of the “eigenvector solution” \( (x(t), y(t)) = e^{\lambda_1 t} v_1 \). This gives you two different real solutions to the differential equation.

d) Find the solution \( (x(t), y(t)) \) with initial value \( (x(0), y(0)) = (1, 0) \).
5. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

a) For each of the following vectors, plot $v$, $Av$, and $A^2v$:
   (1) $v = (1, 0)$
   (2) $v = (0, 1)$
   (3) $v = (1, 1)$

b) For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v, A^{-1}v, v, Av, A^2v, \ldots$.

c) For the vector $v = (1, 1)$, what direction is the vector $A^n v$ approximately pointing when $n$ is very large? In other words, what unit vector does $\frac{A^n v}{\|A^n v\|}$ approximate when $n$ is very large?

d) For the vector $v = (1, 1)$, what direction is $A^{-n}v$ approximately pointing when $n$ is very large?