Math 218D Problem Session: Week 11

Answer Key

1. Matrices with complex eigenvalues

Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) Compute the eigenvalues of A and B. Write each eigenvalue in polar coordinates $z = re^{i\theta}$.

b) Compute the eigenvectors of A and B.

Solution.

a) The eigenvalues of A are $1/2i = 1/2e^{\pi/2i}$ and $-1/2i = 1/2e^{-\pi/2i}$. These eigenvalues have $|\lambda| = 1/2$ and angle $\theta = \pm\pi/2$.

The eigenvalues of B are $(1+i) = \sqrt{2}e^{\pi/4i}$ and $(1-i) = \sqrt{2}e^{-\pi/4i}$.

b) For the matrix A, an eigenvector of $(1/2)i$ is $(1, -i)$, and an eigenvector for $-(1/2)i$ is $(1, i)$.

For the matrix B, an eigenvector $v_1 = (x_1, x_2)$ of B for the eigenvalue $\lambda_1 = (1+i)$ is a solution to $-ix_1 + x_2 = 0$, so we'll use $v_1 = (1, i)$ as the eigenvector.

An eigenvector for $\lambda_2 = \overline{\lambda_1}$ is $v_2 = \overline{v_1} = (1, -i)$.

2. Some quick matrix exponentials

Compute the matrix exponential e^A of:

(1) $A=\begin{pmatrix}2 & 0\\ 0 & -3\end{pmatrix}$,

(2) $A=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$,

(3) $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$,

(4) $A=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$,

(5) $A=\begin{pmatrix}3 & 1 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{pmatrix}$.

Solution.

(1) $A=\begin{pmatrix}2 & 0\\ 0 & -3\end{pmatrix}$, $e^A=\begin{pmatrix}e^2 & 0\\ 0 & e^{-3}\end{pmatrix}$.

(2) $A=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$. A has diagonalization $A=\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}^{-1}$, so $e^A=e^{CDC^{-1}}=Ce^DC^{-1}=C\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}C^{-1}$, which you can multiply to get the final answer.

(3) $A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $e^A=I+A+A^2/2+\cdots=I+A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$.

(4) $A=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$, $e^A=e^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}}=e^{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\cdot e^{\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}}=\begin{pmatrix}e & 0\\ 0 & e\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}=\begin{pmatrix}e & 2e\\ 0 & e\end{pmatrix}$.

(5) $A=\begin{pmatrix}3 & 1 & 0\\ 0 & 3 & 1\\ 0 & 0 & 3\end{pmatrix}$, $e^A=e^{3I}\cdot e^{\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}}=e^3(I+\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}^2/2)=\begin{pmatrix}e^3 & e^3 & e^3/2\\ 0 & e^3 & e^3\\ 0 & 0 & e^3\end{pmatrix}$.

3. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(t)$$

a) Write this as a matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

What is the matrix A?

b) For this matrix A, find the eigenvalues λ_1 and λ_2, as well as the eigenvectors w_1 and w_2.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, which scalars a_1 and a_2 should you choose?

d) Plug the solution with initial value $(x(0), y(0)) = (1, 1)$ to the differential equation, and confirm that it is a solution.

e) For the solution you found in **c)**, compute $(x(1), y(1))$.

Solution.

a) The matrix A in the matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

is $A = \begin{pmatrix} 3 & 2 \\ 4 & -4 \end{pmatrix}$.

b) This matrix A has characteristic polynomial $\lambda^2 + \lambda - 20$, with eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 4$. The eigenvectors are $w_1 = (-2, 8)$ and $w_2 = (-1, 2)$.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, your scalars must solve $(1, 1) = a_1 w_1 + a_2 w_2$. You can solve this by solving the system of linear equations $\left(\begin{array}{cc|c} -2 & -1 & 1 \\ 8 & 2 & 1 \end{array}\right)$. This has solution $a_1 = -5/4, a_2 = 3/2$, i.e. $-5/4(-2, 8) + 3/2(-1, 2) = (1, 1)$.

d) The solution is $u(t) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. When we plug this into the differential equation, we get $u'(t) = a_1 \lambda_1 e^{\lambda_1 t} w_1 + a_2 \lambda_2 e^{\lambda_2 t} w_2$ on one side, and $Au(t) = a_1 e^{\lambda_1 t}(\lambda_1 w_1) + a_2 \lambda_2 e^{\lambda_2 t}(\lambda_2 w_2)$ on the other. Since these are equal, $u(t)$ solves the differential equation. (We didn't actually need to use the values for a_1 and a_2 to check this.)

e) The value of $(x(1), y(1))$ is $-5/4e^{\lambda_1}w_1+3/2e^{\lambda_2}w_2 = (5/2e^{-5}-3/2e^4, -10e^{-5}+3e^4)$. You don't need to simplify any further than this.

4. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) Write this as a matrix differential equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

b) Compute the eigenvalues $\lambda_1,\ \lambda_2$ and eigenvectors $v_1,\ v_2$ of the matrix A.

c) Compute the real and imaginary parts of the "eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$. This gives you two different *real* solutions to the differential equation.

d) Find the solution $(x(t), y(t))$ with initial value $(x(0), y(0)) = (1, 0)$.

Solution.

a) The matrix differential equation would be

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

b) The characteristic polynomial is $(\lambda - 1)^2 + 1 = 0$, which gives the eigenvalues $\lambda_1 = 1 + i$ and $\lambda_1 = 1 - i$. The associated eigenvectors are $w_1 = (i, 1)$ and $w_2 = (-i, 1)$.

c) The eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$ is

$$e^{t(1+i)} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \left(\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right),$$

from which we can observe the real and imaginary part.

d) We write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Hence the solution would be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{i}{2} e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} e^{(1-i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

5. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

a) For each of the following vectors, plot v, Av, and A^2v:

(1) $v = (1,0)$

(2) $v = (0,1)$

(3) $v = (1,1)$

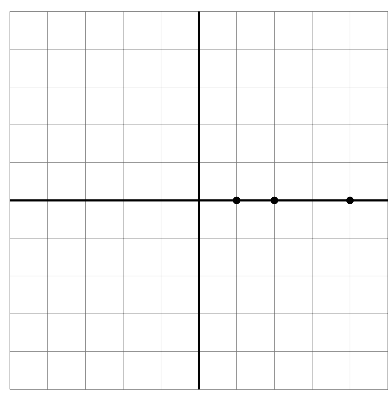
b) For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v,\ A^{-1}v,\ v,\ Av,\ A^2v, \ldots$.

c) For the vector $v = (1,1)$, what direction is the vector $A^n v$ approximately pointing when n is very large? In other words, what unit vector does $\frac{A^n v}{||A^n v||}$ approximate when n is very large?

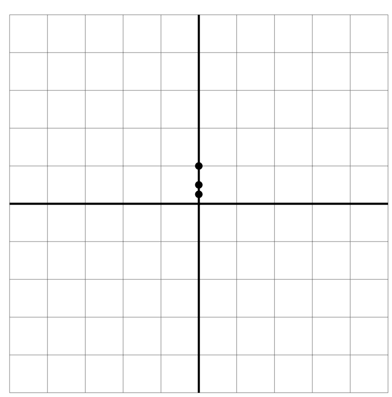
d) For the vector $v = (1,1)$, what direction is $A^{-n}v$ approximately pointing when n is very large?

Solution.

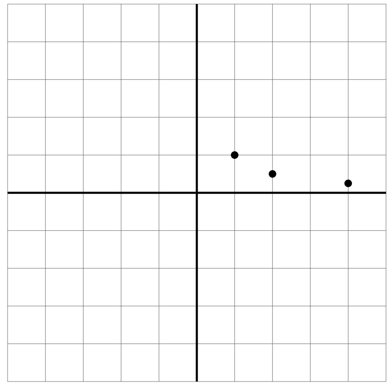
a) (1) $v = (1,0)$



(2) $v = (0,1)$



(3) $v = (1,1)$



b) We'll draw all the shapes on the same plot:

c) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches ∞, is $(1,0)$. We can see this from the picture, but can also compute this using limits. First,

$$\frac{A^n v}{||A^n v||} = \frac{2^n(1,0) + 2^{-n}(0,1)}{\sqrt{2^{2n} + 2^{-2n}}} = \frac{(1,0) + 2^{-2n}(0,1)}{\sqrt{1 + 2^{-4n}}},$$

where the second equality comes from dividing both the numerator and denominator by 2^n. Since $\lim_{n\to\infty}(1,0)+2^{-2n}(0,1) = (1,0)$ and $\lim_{n\to\infty}\sqrt{1+2^{-4n}} = 1$, we find that

$$\lim_{n\to\infty} \frac{A^n v}{||A^n v||} = \frac{(1,0)}{1} = (1,0).$$

d) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches $-\infty$, is $(0,1)$.