Math 218D Problem Session: Week 1

Answer Key

1. Row Echelon Form.

For each of the following linear systems/augmented matrices, do the following:

(1) If it is a linear system, convert it to an augmented matrix. If it is an augmented matrix, convert it to a linear system.

(2) Decide whether or not the augmented matrix is in Row Echelon Form (REF). If it is in REF, circle the pivots/pivot entries. If it is not, explain why not.

a) $$\begin{aligned} x + 2y &= 1 \\ 3y &= 2 \end{aligned}$$

b) $$\left(\begin{array}{cc|c} 5 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{array}\right)$$

c) $$\begin{aligned} 2x + y - z &= 1 \\ 3y + z &= 2 \\ 2x &= 1 \end{aligned}$$

d) $$\left(\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 5 \end{array}\right)$$

e) $$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 4 & 1 \end{array}\right)$$

f) $$\left(\begin{array}{cc|c} 5 & 5 & 5 \\ 1 & 1 & 1 \end{array}\right)$$

Solution.

a) In REF; pivots are 1 and 3.

b) In REF; pivots are 5 and 6.

c) Not in REF.

d) In REF; pivots are 2 and 1.

e) Not in REF.

f) Not in REF.

2. Two Equations and Two Unknowns.
Consider the system of 2 linear equations:

$$\begin{aligned} x - y &= 2 \\ 2x - 4y &= -4. \end{aligned}$$

a) Draw the two lines in $\mathbf{R}^2$ determined by these two equations.

Now, solve the linear system using the following steps:

b) Use one row operation to eliminate x from the second equation.

c) Use another row operation to make the second equation into $y = (?)$.

d) Use a third row operation to make the first equation into $x = (?)$. What is the solution (x, y)?

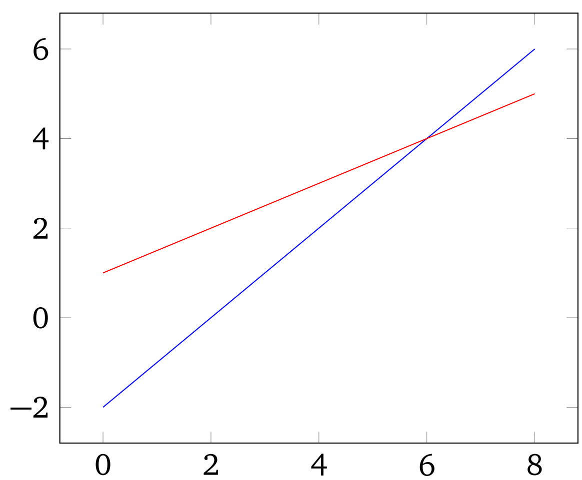
e) Check your answer by plugging your solution into the original equations.

The first row operation is an *elimination* step, while the third is a *substitution* step.

f) After which row operation is the system in REF? Circle the pivot entries.

Solution.

a)



b) The linear system is

$$\begin{aligned} x - y &= 2 \\ 2x - 4y &= -4. \end{aligned}$$

Subtract $2 \cdot R_1$ from R_2 to obtain:

$$\begin{aligned} x - y &= 2 \\ -2y &= -8. \end{aligned}$$

c) Divide the second row by -2 to obtain:

$$\begin{aligned} x - y &= 2 \\ y &= 4. \end{aligned}$$

d) Add the second row to the first row to obtain:

$$\begin{aligned} x &= 6 \\ y &= 4. \end{aligned}$$

This is the solution.

e) $6-4=2,\ 2\cdot 6-4\cdot 4=-4.$

f) The system first becomes in REF after the 1st row operation. The pivots are 1 and -2.

3. Three Equations Three Unknowns.

Consider the system of three linear equations

$$\begin{aligned} x_1 - 3x_2 + \quad x_3 &= \ \ 4 \\ 2x_1 - 8x_2 + \ 8x_3 &= -2 \\ -6x_1 + 3x_2 - 15x_3 &= \ \ 9. \end{aligned}$$

a) Convert this linear system into a **matrix equation** $Ax = b$, where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

b) Write down the augmented matrix $\left(A \mid b\right)$.

c) Use elementary row operations to convert the augmented matrix into a Row Echelon Form matrix.

Hint: Begin by eliminating x_1: add a multiple of the first row to the second and third rows, so that the 2 and −6 in the first column are replaced with 0. This takes two row operations.

d) How many elementary row operations did you use?

e) Convert the augmented matrix back into a system of three linear equations, and use back-substitution to find the solution vector x.

f) Check your answer by multiplying A by x and confirming that it equals b.

Solution.

a) $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}.$

b) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right).$$

c) First, replace R_2 by $R_2 - 2R_1$ ($R_2 \mathrel{+}= -2R_1$).

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{array}\right).$$

Then $R_3 \mathrel{+}= 6R_1$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array}\right).$$

Now, you can do row scaling here, although you don't need to. Let's do it now to simplify our rows: $R_2 \times= -(1/2)$ and $R_3 \times= -(1/3)$ (combining two elementary operations at once):

$$\left(\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 5 & 3 & -11 \end{array}\right).$$

We do one more row addition, replacing R_2 with $R_2 - 5R_1$ ($R_2 -= 5R_1$):

$$\left(\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 18 & -36 \end{array}\right).$$

Do one more row scaling, replacing R_3 with $\frac{1}{18}R_3$ ($R_3 \times= 1/18$):

$$\left(\begin{array}{rrr|r} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

d) I used 6 elementary row operations, but the row scalings could have been avoided, giving you as few as 3.

e) The system of equations is now

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ x_2 - 3x_3 &= 5 \\ x_3 &= -2 \end{aligned}.$$

Substituting $x_3 = -2$, we obtain the system

$$\begin{aligned} x_1 - 3x_2 &= 6 \\ x_2 &= -1 \\ x_3 &= -2 \end{aligned}.$$

Substituting $x_2 = -1$, we obtain the system

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -1 \\ x_3 &= -2 \end{aligned},$$

which is the solution.

f) Check $\begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}$.

4. Another One—What's Different?

Consider the system of three linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ 2x_1 - 4x_2 + 8x_3 &= 2 \\ x_1 - 3x_2 - x_3 &= 1. \end{aligned}$$

a) Use row operations to eliminate x_1 from the second and third equation.

b) You can now use a single row operation to put the linear system in to REF. What row operation is it?

c) Substitute and solve for (x_1, x_2, x_3). Check your answer with original linear system.

Solution.

a) The linear system is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ 2x_1 - 4x_2 + 8x_3 &= 2 \\ x_1 - 3x_2 - x_3 &= 1. \end{aligned}$$

By doing two row subtraction operations ($R_2 -= 2R_1$ and $R_3 -= R_1$), we obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ 6x_3 &= 6 \\ -x_2 - 2x_3 &= 3. \end{aligned}$$

b) We swap rows 2 and 3 to obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ -x_2 - 2x_3 &= 3 \\ 6x_3 &= 6. \end{aligned}$$

c) Dividing row 3 by 6 gives $x_3 = 1$, which we substitute into the first two equations:

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ -x_2 &= 5 \\ x_3 &= 1. \end{aligned}$$

Dividing row 2 by -1 gives $x_2 = -5$, which we substitute into the 1st equation:

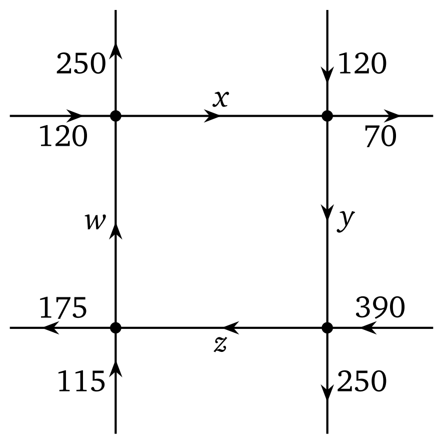
$$\begin{aligned} x_1 &= -13 \\ x_2 &= -5 \\ x_3 &= 1. \end{aligned}$$

This is the solution.

5. Traffic Jam.

You should have seen this traffic example in lecture, although the numbers were a bit different. We'll now will explore it in more detail.

Traffic flow (cars/hr)



This represents a town with 4 main roads, as well as 8 roads in and out of town. Each road is one-way, in the direction indicated by the arrows. The 8 roads have a fixed number of cars/hour which travel in and out of town on them.

Question: How many cars/hour travel on each of the 4 main roads?

We'll use linear algebra to answer this question. At each intersection, the number of incoming cars per hour must equal the number of outgoing cars. This gives 4 linear equations, in the variables x, y, z, w:

$$\begin{aligned} 120 + w &= 250 + x \\ 120 + x &= 70 + y \\ 390 + y &= 250 + z \\ 115 + z &= 175 + w. \end{aligned}$$

a) Rewrite these equations with the variables on the left and the numbers on the right. Write the variables on the left side in neat columns.

b) Use the first equation to eliminate x from the equations below it.

c) Use the second equation to eliminate y from the equations below it.

d) Use the third equation to eliminate z from the equations below it.

Remember: after each step, you should have a linear system with 4 equations and 4 variables.

e) At this point you are done with elimination. Can you substitute to solve the system? Can you answer the original **Question**?

f) How many solutions does this system have? Can you explain why from the original picture?

g) Convert the linear system obtained after step 3 into an augmented matrix. It should be in REF. Circle the pivot entries. Does each row have a pivot entry? Does each column to the left of the augmentation line have a pivot entry?

Solution.

a) We start with

$$\begin{aligned} 120 + w &= 250 + x \\ 120 + x &= \ \ 70 + y \\ 390 + y &= 250 + z \\ 115 + z &= 175 + w \end{aligned}$$

or

$$\begin{aligned} x \qquad\quad - w &= -130 \\ -x + y \qquad\quad &= \ \ 50 \\ -y + z \quad &= \ \ 140 \\ -z + w &= \ \ -60. \end{aligned}$$

b) Eliminating x from the second equation gives

$$\begin{aligned} x \qquad\quad - w &= -130 \\ y \quad - w &= -80 \\ -y + z \quad &= 140 \\ -z + w &= -60. \end{aligned}$$

c) Eliminating y from the third equation gives

$$\begin{aligned} x \qquad\quad - w &= -130 \\ y \quad - w &= -80 \\ z - w &= 60 \\ -z + w &= -60. \end{aligned}$$

d) Eliminating z from the fourth equation gives

$$\begin{aligned} x \qquad\quad - w &= -130 \\ y \quad - w &= -80 \\ z - w &= 60 \\ 0 &= 0. \end{aligned}$$

e) We can't just use substitution, as our final equation is not of the form $w = (?)$. The number of cars on roads x, y, and z all depend on how many cars are on w.

f) The system has infinitely many solutions. There can be as many cars as you want, travelling in a circle around the town.

g) The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 0 & 1 & 0 & -1 & -80 \\ 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The pivots are the 1's. Not every row has a pivot. The fourth column does not have a pivot—as we will discuss in week 3, this means that we can find a solution which makes the fourth variable take any value we want. Such a variable is called a *free variable*.