Solving Systems of Equations using Elimination

Here’s a system of 3 equations in 3 variables:

\[
\begin{align*}
   x_1 + 2x_2 + 3x_3 &= 6 \\
   2x_1 - 3x_2 + 2x_3 &= 14 \\
   3x_1 + x_2 - x_3 &= -2
\end{align*}
\]

How to solve it?

- **Substitution**: solve 1st equation for \(x_1\), substitute into 2nd & 3rd, continue.
- **Elimination**: “combine” the equations to eliminate variables.

Elimination turns out to scale much better (to more equations & variables), so we’ll focus on that.

"replace the 2nd equation with the 2nd minus 2x the 1st"

**Eg:**

\[
\begin{align*}
   x_1 + 2x_2 + 3x_3 &= 6 \\
   2x_1 - 3x_2 + 2x_3 &= 14 \\
   3x_1 + x_2 - x_3 &= -2
\end{align*}
\]

\[
\begin{align*}
R_2 &= 2R_1 \\
   x_1 + 2x_2 + 3x_3 &= 6 \\
   2x_1 - 3x_2 + 2x_3 &= 14 \\
   R_3 &= 3R_1 \\
   3x_1 + x_2 - x_3 &= -2
\end{align*}
\]

Now we have eliminated \(x_1\) from the 2nd & 3rd eqs.
These now form 2 equations in 2 variables: simplify:

\[ x_1 + 2x_2 + 3x_3 = 6 \]
\[ -7x_2 - 4x_3 = 2 \]
\[ -5x_2 - 10x_3 = -20 \]

We eliminated \( x_2 \) from the last equation: now it's one equation in one variable. Easy!

We can now solve via back-substitution:

\[ -\frac{50}{7} \times 3 = -\frac{150}{7} \Rightarrow x_3 = 3. \]

Substitute into 2nd equation:

\[ -7x_2 - 4x_3 = 2 \Rightarrow -7x_2 - 4 \cdot 3 = 2 \]

Now solve for \( x_2 \):

\[ -7x_2 - 12 = 2 \Rightarrow -7x_2 = 14 \Rightarrow x_2 = -2 \]

Substitute both into 1st equation:

\[ x_1 + 2x_2 + 3x_3 = 6 \Rightarrow x_1 + 2 \cdot (-2) + 3 \cdot 3 = 6 \]

Now solve for \( x_1 \):

\[ x_1 - 4 + 9 = 6 \Rightarrow x_1 = 1 \]

Check:

\[ 1 + 2 \cdot (-2) + 3 \cdot (3) = 6 \]
\[ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot (3) = 14 \]
\[ 3 \cdot 1 + (-2) - 3 = -2 \]

NB: In this case there was one solution — since we could
isolate each variable, all values were determined.

Does this always work?

Eg: \[ 4x_1 + 3x_3 = 2 \]
\[ x_1 + x_2 - x_3 = 3 \]
\[ 2x_1 - 3x_2 - 6x_3 = -3 \]

\[ R_1 \leftarrow R_1 \]
\[ x_1 + x_2 - x_3 = 3 \]
\[ 4x_1 + 3x_3 = 2 \]
\[ 2x_1 - 3x_2 - 6x_3 = -3 \]

Now eliminate as before:

\[ R_2 \leftarrow 2R_1 \]
\[ x_1 + x_2 - x_3 = 3 \]
\[ 4x_1 + 3x_3 = 2 \]
\[ -5x_2 - 4x_3 = -9 \]

\[ R_3 \leftarrow \frac{5}{4} R_2 \]
\[ x_1 + x_2 - x_3 = 3 \]
\[ 4x_1 + 3x_3 = 2 \]
\[ -\frac{1}{4}x_3 = -\frac{13}{2} \]

Solve using back-substitution:

\[ -\frac{1}{4}x_3 = -\frac{13}{2} \Rightarrow x_3 = 26 \]

Substitute into 2nd equation:

\[ 4x_1 + 3(26) = 2 \Rightarrow x_2 = -19 \]

Substitute both into 1st equation:

\[ x_1 - 19 - 26 = 3 \Rightarrow x_1 = 48 \]

NB again, there is one solution.
Check: \[ 4x_1 + 3x_3 = 2 \quad \Rightarrow \quad 4(-19) + 3(26) = 2 \]
\[ x_1 + x_2 - x_3 = 3 \quad \Rightarrow \quad 48 - 19 - 26 = 3 \]
\[ 2x_1 - 3x_2 - 6x_3 = -3 \quad \Rightarrow \quad 2(48) - 3(-19) - 6(26) = -3 \]

Eg: \[ x_1 + 2x_2 + 3x_3 = 1 \]
\[ 4x_1 + 5x_2 + 6x_3 = 0 \]
\[ 7x_1 + 8x_2 + 9x_3 = -1 \]

\[ R_2 = 4R_1 \]
\[ x_1 + 2x_2 + 3x_3 = 1 \]
\[ 4x_1 + 5x_2 + 6x_3 = 0 \]
\[ 7x_1 + 8x_2 + 9x_3 = -1 \]

\[ R_3 = 7R_1 \]
\[ -3x_2 - 6x_3 = -4 \]
\[ -6x_2 - 12x_3 = -8 \]

\[ R_3 = 2R_2 \]
\[ x_1 + 2x_2 + 3x_3 = 1 \]
\[ -3x_2 - 6x_3 = -4 \]
\[ 0 = 0 \]

Are we done? Yes: choose any value for \( x_3 \), then back-substitute to find \( x_1, x_2 \):
\[ -3x_2 = -4 + 6x_3 \quad \Rightarrow \quad x_2 = \frac{4}{3} - 2x_3 \]
\[ x_1 = 1 - 2x_2 - 3x_3 = 1 - \frac{8}{3} + 4x_3 - 3x_3 \]
\[ x_1 = -\frac{5}{3} + x_3 \]

Eg: \( x_3 = 1 \quad \Rightarrow \quad x_1 = -\frac{2}{3}, \ x_2 = -\frac{2}{3} \)

Check: \[ -\frac{2}{3} - 4\frac{1}{3} + 3 = 1 \]
\[ -8\frac{1}{3} - 10\frac{1}{3} + 6 = 0 \]
\[ -14\frac{1}{3} - 16\frac{1}{3} + 9 = -1 \]

In this case there are infinitely many solutions.
We'll deal with this in Week 3.
If our original equations were true, then $0 = 1$. Thus our system has no solutions. (last 2 eqns are parallel planes)

**Row Operations** are the allowed manipulations we can perform on our equations.

1. \[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 6 \\
    2x_1 - 3x_2 + 2x_3 &= 14 \\
    3x_1 + x_2 - x_3 &= -2
\end{align*}
\]
   \[\overset{R_2 = 2R_1}{\longrightarrow}\]
   \[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 6 \\
    -7x_2 - 4x_3 &= 2 \\
    3x_1 + x_2 - x_3 &= -2
\end{align*}
\]
   \[\overset{\text{row replacement}}{\longrightarrow}\]
   \[\overset{\text{replace } R_2 \text{ by } R_2 - 2R_1}{\longrightarrow}\]

2. \[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 6 \\
    2x_1 - 3x_2 + 2x_3 &= 14 \\
    3x_1 + x_2 - x_3 &= -2
\end{align*}
\]
   \[\overset{R_1 \leftrightarrow R_2}{\longrightarrow}\]
   \[
\begin{align*}
    2x_1 - 3x_2 + 2x_3 &= 14 \\
    x_1 + 2x_2 + 3x_3 &= 6 \\
    3x_1 + x_2 - x_3 &= -2
\end{align*}
\]
   \[\overset{\text{row swap}}{\longrightarrow}\]
   \[\overset{\text{(change order)}}{\longrightarrow}\]

**Eg:** \[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 1 \\
    4x_1 + 5x_2 + 6x_3 &= 0 \\
    7x_1 + 8x_2 + 9x_3 &= 0
\end{align*}
\]
(3) \[ \begin{align*}
3x_1 + 2x_2 + 3x_3 &= 6 \\
2x_1 - 3x_2 + 2x_3 &= 14 \\
3x_1 + x_2 - x_3 &= -2
\end{align*} \]

scalar multiplication (by nonzero scalar)

Obviously if \((x_0, x_0, x_0)\) is a solution before doing a row operation, then it is true after. E.g. row replacement:

\[
\begin{align*}
x_1 + 2x_2 + 3x_3 &\rightarrow 6 = 6 \\
2x_1 - 3x_2 + 2x_3 &\rightarrow 14 = 14
\end{align*}
\]

Fact: All these operations are reversible: if you have a solution \((x_0, x_0, x_0)\) after doing a row operation, then it's also a solution before.

This was the whole point: we wanted to solve our original system of equations!

Questions: How do you undo (reverse):

- \(R_i \rightarrow R_j\) ?
- \(R_i x = 2\) ?
- \(R_i \rightarrow R_2\) ?
- \(R_i \leftarrow R_2\) ?
The variables \( x_0, x_2, \ldots \) are just placeholders; only their coefficients matter. Let’s extract them into a matrix.

Three Ways to Write System of Linear Equations

1) As a system of equations:
\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 6 \\
    2x_1 - 3x_2 + 2x_3 &= 14
\end{align*}
\]

2) As a matrix equation \( Ax = b \)
\[
\begin{bmatrix}
    1 & 2 & 3 \\
    2 & -3 & 2
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
= 
\begin{bmatrix}
    6 \\
    14
\end{bmatrix}
\]

If you expand out the product you get
\[
\begin{align*}
    \begin{bmatrix}
        x_1 + 2x_2 + 3x_3 \\
        2x_1 - 3x_2 + 2x_3
    \end{bmatrix}
    &= 
\begin{bmatrix}
    6 \\
    14
\end{bmatrix}
\end{align*}
\]

which is what we had before.

The coefficient matrix \( A \) comes from the coefficients of the variables:
\[
\begin{bmatrix}
    1 & 2 & 3 \\
    2 & -3 & 2
\end{bmatrix}
\overset{\text{coefficients}}{\leftrightarrow}
\begin{bmatrix}
    1x_1 + 2x_2 + 3x_3 \\
    2x_1 - 3x_2 + 2x_3
\end{bmatrix}
\]
The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ contains the unknowns or variables.

**NB:** $A$ is an $m \times n$ matrix where
- $m = \#$ equations
- $n = \#$ variables
- $b \in \mathbb{R}^m$
- $x \in \mathbb{R}^n$

(3) As an augmented matrix.
This is a notational convenience: just squash $A$ & $b$ together and separate with a line.

\[
\begin{bmatrix}
1 & 2 & 3 & | & 6 \\
2 & -3 & 2 & | & 14 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1x_1 + 2x_2 + 3x_3 = 6 \\
2x_1 - 3x_2 + 2x_3 = 14 \\
\end{bmatrix}
\]

\[
[A \mid b]
\]

Augmented matrices are good for row operations, which only affect the coefficients (not the variables):

\[
x_1 + 2x_2 + 3x_3 = 6 \quad R_2 \rightarrow 2R_1 \quad x_1 + 2x_2 + 3x_3 = 6
\]
\[
2x_1 - 3x_2 + 2x_3 = 14 \quad -7x_2 - 4x_3 = 2
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 6 \\
2 & -3 & 2 & | & 14 \\
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 3 & | & 6 \\
0 & -7 & -4 & | & 2 \\
\end{bmatrix}
\]
Let's solve the system from before using augmented matrices:

\[
\begin{align*}
&x_1 + 2x_2 + 3x_3 = 6 \\
&2x_1 - 3x_2 + 2x_3 = 14 \\
&3x_1 + x_2 - x_3 = -2
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 6 \\
2 & -3 & 2 & | & 14 \\
3 & 1 & -1 & | & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 6 \\
0 & -7 & 4 & | & 2 \\
3 & 1 & -1 & | & -2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 6 \\
0 & -7 & 4 & | & 2 \\
0 & 0 & -\frac{30}{7} & | & -\frac{150}{7}
\end{bmatrix}
\]

\[
\begin{align*}
&x_1 + 2x_2 + 3x_3 = 6 \\
&-7x_2 - 4x_3 = 2 \\
&-\frac{30}{7}x_3 = -\frac{150}{7}
\end{align*}
\]

Now use back-substitution like before.
What does it mean to be “done”? 
(in terms of augmented matrices)

**Def:** A matrix is in row echelon form (REF) if:
1. The first nonzero entry of each row is to the right of the row above it.
2. All zero rows are at the bottom.

\[
\begin{bmatrix}
  \bullet & \bullet & \bullet & \bullet \\
  0 & 0 & \bullet & \bullet \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

- \( \bullet = \text{nonzero} \)
- \( \bullet = \text{anything} \)

**REF:**
\[
\begin{bmatrix}
  1 & 2 & -1 & 4 \\
  0 & 0 & 3 & 12 \\
\end{bmatrix}
\]

**Not REF:**
\[
\begin{bmatrix}
  2 & 0 & 1 & 0 \\
\end{bmatrix}
\]

Important: When checking if an augmented matrix is in REF, ignore the augmentation line.

\[
\begin{bmatrix}
  1 & 2 & -1 & 4 \\
  0 & 0 & 3 & 12 \\
\end{bmatrix}
\]

REF? \[
\begin{bmatrix}
  1 & 2 & -1 & 4 \\
  0 & 0 & 3 & 12 \\
\end{bmatrix}
\]

\[\text{\checkmark}\]
Upshot: The elimination procedure terminates when your (augmented) matrix is in REF.

Def: The pivot positions (pivots) of a matrix are the positions of the 1st nonzero entries of each row after you put it into REF.

\[
\begin{bmatrix}
1 & -1 & 4 \\
0 & 0 & 3 & 12
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 6 \\
0 & -7 & -\frac{80}{7} & -\frac{150}{7}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -3 & -6 & -4
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -3 & -6 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\(\circ\) = pivots

Remarkably, this is well-defined!
The rank of a matrix is the number of pivots it has (in REF).

Example:

\[
\begin{bmatrix}
1 & 2 & 3 & 6 \\
2 & -3 & 2 & 14 \\
3 & 1 & -1 & -2
\end{bmatrix}
\xrightarrow{\text{REF}}
\begin{bmatrix}
1 & 2 & 3 & 6 \\
0 & -7 & -4 & 2 \\
0 & 0 & -\frac{300}{7} & -\frac{150}{7}
\end{bmatrix}
\]

rank = 3

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
4 & 5 & 6 & 0 \\
7 & 8 & 9 & -1
\end{bmatrix}
\xrightarrow{\text{REF}}
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -3 & -6 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

rank = 2
**Number of Solutions (in terms of pivots)**

The most basic question you can ask about a system of equations is: how many solutions does it have? This is entirely determined by the pivot positions/pivot columns (columns with a pivot).

1) The system

\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & -7 & 4 \\
0 & 0 & -\frac{8}{11}
\end{bmatrix}
\begin{bmatrix}
6 \\
2 \\
-\frac{250}{7}
\end{bmatrix}
\]

had one solution. It has a pivot in every column except the augmented column. This means every variable will be isolated when doing back-substitution.

2) The system

\[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 3 & 6 & 10 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

had no solutions. It has a pivot in the augmented column, which leads to the equation 0=1.
The system

\[
\begin{bmatrix}
1 & 2 & 3 & 1 \\
0 & -3 & 6 & -4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (p. 4)

had infinitely many solutions. It has no pivot in the augmented column and no pivot in the column for the variable \( x_3 \). You can't isolate \( x_3 \), so you can choose any value.

**NB:** You have to put the system in REF to find its pivots, so you have to do work to know how many solutions there are all.

**Def:** A system is **consistent** if it has at least 1 solution (so 1 or \( \infty \)). It is **inconsistent** otherwise.