The Basis Theorem
Recall from last time:

\[ \text{Basis of } \mathbb{R}^n = \text{cols of an invertible } n \times n \text{ matrix} \]

For an \( n \times n \) matrix,

\[ \text{full col rank } \iff \text{invertible } \iff \text{full row rank} \]

In terms of columns, \( n \) vectors in \( \mathbb{R}^n \)

\[ \text{spans } \mathbb{R}^n \iff \text{linearly independent} \]

This is a special case of the basis theorem.

Basis Theorem: Let \( V \) be a subspace of \( \mathbb{R}^d \)

1. If \( d \) vectors span \( V \) then they're a basis
2. If \( d \) vectors in \( V \) are LI then they're a basis.

So if you have the correct number of vectors, you only need to check one of spans/LI.

Eg:

• Two noncollinear vectors in a plane
  form a basis.
• Two vectors that span a plane form a basis.

This is how the Basis Thm makes our intuition precise.
Geometry of Dot Products

We are now aiming to find the "best" approximate solution of $Ax = b$ when no actual solution exists.

Eg: find the best-fit ellipse through these points from the 1st lecture...

Q: How close can $Ax$ get to $b$?

$\text{Col}(A) = \{ Ax : x \in \mathbb{R}^n \}$

so this means: what is the closest vector $b$ in $\text{Col}(A)$ to $b$?

A: $b - b$ is perpendicular to $\text{Col}(A)$

So we want to understand what vectors are perpendicular to a subspace.

We will study the geometric notion of "perpendicular" using the algebra of dot products.

Recall: $v = (x_1, \ldots, x_n)$, $w = (y_1, \ldots, y_n)$ \Rightarrow $v \cdot w = x_1 y_1 + \cdots + x_n y_n = v^T w$

$v^T w = (x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_1 y_1 + \cdots + x_n y_n) = \langle v, w \rangle$
Dot products measure *length* & *angles* \((\text{eg. } 90^\circ)\)

→ geometric questions about *length* & *angles*
become algebraic questions about dot products

**Recall:** If \(v=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\), then
\[

\|v\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0

\]

**Def:** The length of \(v\) is
\[

\|v\| = \sqrt{v \cdot v} \quad \text{i.e.} \quad \|v\|^2 = v \cdot v

\]

This makes sense by the Pythagorean theorem: \(v=(3,4)\)

**Sanity Check:** \(c \in \mathbb{R}, \ v \in \mathbb{R}^n\)
\[

\|cv\| = \|c(x_1, \ldots, x_n)\| = \|\begin{pmatrix} c \ x_1 \\ \vdots \\ c \ x_n \end{pmatrix}\| = \sqrt{(cx_1)^2 + \cdots + (cx_n)^2}

= |c| \cdot \sqrt{x_1^2 + \cdots + x_n^2} = |c| \cdot \|v\| \quad \checkmark

\]

\[

\|cv\| = |c| \cdot \|v\|

\]

\[\text{Eq: } 2v \text{ is twice as long as } v.\]
So \(\text{is } -2v.\)
Def: The distance from \( v \) to \( w \) is \( \|v-w\| = \|w-v\| \)

Def: A unit vector is a vector of length 1, ie \( \|v\| = 1 \), ie. \( \|v\|^2 = v \cdot v = 1 \)

If \( v = (x_1, \ldots, x_n) \), then \( v \) is a unit vector
\[ \iff \quad x_1^2 + \cdots + x_n^2 = 1 \]
\[ \iff \quad v \text{ lies on the unit } (n-1) \text{-sphere} \]
\[ \text{ (} n=2: \text{ unit circle) } \]

If \( v \neq 0 \), the unit vector in the direction of \( v \) is the vector
\[ u = \frac{1}{\|v\|} \cdot v = \frac{v}{\|v\|} \quad \text{ (scalar \times vector)} \]

NB: \( \|u\| = \left| \frac{1}{\|v\|} \right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1 \)
Eq: $\mathbf{v} = \binom{4}{3}$  $\|\mathbf{v}\| = \sqrt{3^2+4^2} = 5$

$\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \frac{1}{5} \binom{4}{3} = \binom{4/5}{3/5}$

NB: all unit vectors in \( \mathbb{R}^2 \) are on the unit circle.

What about \( \mathbf{v} \cdot \mathbf{w} \) for \( \mathbf{v} \neq \mathbf{w} \)?

Law of Cosines:

\[ c^2 = a^2 + b^2 - 2ab \cos \theta \]

Vector Version:

\[ \|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos \theta \]

(a = \|\mathbf{v}\|  b = \|\mathbf{w}\|  c = \|\mathbf{v} - \mathbf{w}\|)

Algebra: "left hand side" LHS: $\|\mathbf{v} - \mathbf{w}\|^2 : = (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w})$

Foil

$= \mathbf{v} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w} - 2\mathbf{v} \cdot \mathbf{w}$

$= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos \theta$

"right hand side" RHS: $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos \theta$

$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta$ or $\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$ (if \( \mathbf{v}, \mathbf{w} \neq 0 \))
Def: The angle from \( v \) to \( \omega \) \((v, \omega \neq \theta)\) is
\[
\theta = \cos^{-1}\left(\frac{v \cdot \omega}{\|v\|\|\omega\|}\right)
\]

NB: \( \cos\theta = \left|\frac{v \cdot \omega}{\|v\|\|\omega\|}\right| \in [0, 1] \)
\[
\Rightarrow \|v \cdot \omega\| \leq \|v\|\|\omega\|
\]

Schwartz Inequality: \( |v \cdot \omega| \leq \|v\|\|\omega\| \)

Def: Vectors \( v \) and \( \omega \) are orthogonal or perpendicular, written \( v \perp \omega \), if \( v \cdot \omega = 0 \)

This says that either:
- \( v = 0 \) or \( \omega = 0 \) (or both), or
- \( \cos(\theta) = 0 \iff \theta = \pm 90^\circ \)

NB: The zero vector is orthogonal to every vector, \( \theta \cdot v = 0 \) for all \( v \)
Orthogonality

We want to know: "which vectors are \( \perp \) a subspace?"
Let's start with: "which vectors are \( \perp \) some vector?"

Eg: Find all vectors orthogonal to \( v=(1) \).

We need to solve \( v \cdot x = 0 \)

\[ \iff v^T x = 0 \]

This is just \( \text{Null}(v^T) \):

\[
\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \rightarrow x_1 + x_2 + x_3 = 0
\]

\[ x_1 = -x_2 - x_3 \]
\[ x_2 = x_2 \]
\[ x_3 = x_3 \]

PVP

\[ x = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \]

[Demo]

\[ \rightarrow \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \]

$\text{a plane}$

Check: \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \cdot (1) = 0 \)
\( \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cdot (1) = 0 \)

$\checkmark$
Eq: Find all vectors orthogonal to $v_1 = (1) \& v_2 = (1,0)$

We need to solve
\[
\begin{cases} 
    v_1^T \cdot x = 0 \Rightarrow x_1 + x_2 + x_3 = 0 \\
    v_2^T \cdot x = 0 \Rightarrow x_1 + x_2 = 0
\end{cases}
\]

Equivalently,
\[
(-v_1^T - v_2^T) \cdot x = (v_1 \cdot x, v_2 \cdot x) = 0
\]

So we want $\text{Null} \left( -v_1^T - v_2^T \right) = \text{Null} \left( \begin{array}{l}1 \\ 1 \\ 0 \end{array} \right)$

\[
\left( \begin{array}{l}1 \\ 1 \\ 0 \end{array} \right) \xrightarrow{\text{RREF}} \left( \begin{array}{l}1 \\ 0 \\ 0 \end{array} \right)
\]

\[
\text{PF: } x_1 = -x_2 \\
\text{PF: } x_2 = x_2 \\
\text{PF: } x_3 = 0
\]

\[
\text{PF: } x = x_2 \left( \begin{array}{l}1 \\ 0 \\ 0 \end{array} \right)
\]

~\text{Span } \left\{ \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \right\}

a \text{ line}

Check: \[
\left( \begin{array}{l}0 \\ \frac{1}{2} \end{array} \right) \cdot \left( \begin{array}{l}1 \\ 0 \end{array} \right) = 0 \quad \left( \begin{array}{l}0 \\ \frac{1}{2} \end{array} \right) \cdot \left( \begin{array}{l}1 \\ \frac{1}{2} \end{array} \right) = 0
\]

[done]
NB: If \( x \perp v_1 \) and \( x \perp v_2 \) then
\[
x \cdot (av_1 + bv_2) = a \cdot x \cdot v_1 + b \cdot x \cdot v_2 = a \cdot 0 + b \cdot 0 = 0
\]
so \( x \) is orthogonal to every vector in \( \text{Span}\{v_1, v_2\} \)

[Demo again]

More generally,
\[
\{ \begin{array}{ll} v \in \mathbb{R}^n : & \text{v is orthogonal to every vector} \\
\text{in} \ (\text{Span}\{v_1, \ldots, v_n\}) \end{array} \} = \text{Null} \left( \begin{array}{c} -v_1^T \\ \vdots \\ -v_n^T \end{array} \right)
\]

This is awkward to say - let's give it a name.

**Def:** Let \( V \) be a subspace of \( \mathbb{R}^n \).

The *orthogonal complement of \( V \) is*
\[
V^\perp = \{ w \in \mathbb{R}^n : \text{w is orthogonal to every vector in} \ V \} \]

NB: Note the difference in notations:
- \( V^\perp \) is the orthogonal complement of a subspace
- \( A^T \) is the transpose of a matrix
**NB:** If \( x \) is in both \( V \) and \( V^\perp \) then \( x \) is orthogonal to itself:

\[
x \cdot x = 0 \Rightarrow x = 0,
\]
so \( V \cap V^\perp = \{0\} \) intersect.

So we showed above:

\[
\text{Span} \{v_1, \ldots, v_n \} = \text{Nul} \left( \begin{pmatrix} -v_1^T \\ \vdots \\ -v_n^T \end{pmatrix} \right)
\]

**Eg:** \( V = \text{Span} \{v\} \Rightarrow V^\perp = \text{Nul} \{v\} \)

**Eg:** \( V = \text{Span} \{v_1, v_2\} \Rightarrow V^\perp = \text{Nul} \left( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \)

**Eg:** \( \text{so} V^\perp = \mathbb{R}^n \quad (\mathbb{R}^n)^\perp = \{0\} \)

**Fact:** \( V^\perp \) is also a subspace of \( \mathbb{R}^n \).

**Check:**

1. Let \( x, y \in V^\perp \). So \( x \cdot v = 0 \) and \( y \cdot v = 0 \) for every \( v \in V \). So \( (x+y) \cdot v = x \cdot v + y \cdot v = 0 + 0 \) for every \( v \in V \), so \( x+y \in V^\perp \).
(2) Let \( x \in V^\perp \), \( c \in \mathbb{R} \). So \( x \cdot v = 0 \) for every \( v \in V \). So \( (cx) \cdot v = c(x \cdot v) = c \cdot 0 = 0 \) for every \( v \in V \) \( \Rightarrow cx \in V^\perp \).

(3) \( 0 \cdot v = 0 \) for every \( v \in V \) \( \Rightarrow 0 \in V^\perp \).

Or:

Every subspace is a span, and the orthogonal complement of a span is a null space (which is a subspace).

Facts: Let \( V \) be a subspace of \( \mathbb{R}^n \).

(1) \( \dim(V) + \dim(V^\perp) = n \) \([\text{demos}]\)

(2) \( (V^\perp)^\perp = V \)

NB: (2) says \( V \) and \( V^\perp \) are orthogonal complements of each other. Subspaces come in orthogonal complement pairs.
Orthogonality of the Four Subspaces

Recall: If someone gives you a subspace, Step 0 is to write it as a column space or a null space. So we want to understand \( \text{Col}(A)^\perp \) & \( \text{Null}(A)^\perp \).

Let \( A = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \). Then

\[
\text{Col}(A)^\perp = \text{Span}\{v_1, \ldots, v_n\}^\perp = \text{Null} \left( \begin{pmatrix} -v_1^T \\ \vdots \\ -v_n^T \end{pmatrix} \right) = \text{Null}(A^T)
\]

\( \text{Col}(A)^\perp = \text{Null}(A^T) \)

Take \( (\cdot)^\perp \) \( \xrightarrow{\text{Col}(A) = (\text{Col}(A)^\perp)^\perp} = \text{Null}(A^T)^\perp \)

Replace \( A \) by \( A^T \)

\( \text{Row}(A) = \text{Col}(A^T) = \text{Null}(A)^\perp \)

and \( \text{Row}(A)^\perp = \text{Null}(A) \)

Orthogonality of the Four Subspaces:

\[
\begin{align*}
\text{Col}(A)^\perp &= \text{Null}(A^T) \\
\text{Null}(A)^\perp &= \text{Row}(A) \\
\text{Null}(A^T)^\perp &= \text{Col}(A) \\
\text{Row}(A)^\perp &= \text{Null}(A)
\end{align*}
\]
This says the two row picture subspaces \( \text{Row}(A), \text{Null}(A) \) are orthogonal complements, & the two column picture subspaces \( \text{Col}(A), \text{Null}(A^T) \) are orthogonal complements.

\[ V = \{ x \in \mathbb{R}^3 : x + 2y = 0 \} \] Find a basis for \( V^\perp \)

**Step 0:** \( V = \text{Null} \left( \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \right) \rightarrow V^\perp = \text{Row} \left( \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \right) \)

\[ V^\perp = \text{Span} \left\{ \left( \begin{pmatrix} 2 \end{pmatrix}, \left( \begin{pmatrix} 1 \end{pmatrix} \right) \right) \right\} : \text{no elimination needed!} \]

\[ A = \begin{pmatrix} 1 & 2 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 & 0 \end{pmatrix} \]

\( \rightarrow \text{Null}(A) = \text{Span} \left\{ \left( \begin{pmatrix} -2 \end{pmatrix} \right) \right\} \quad \text{Null}(A^T) = \text{Span} \left\{ \left( \begin{pmatrix} -1 \end{pmatrix} \right) \right\} \)

\( \text{Col}(A) = \text{Span} \left\{ \left( \begin{pmatrix} 1 \end{pmatrix} \right) \right\} \quad \text{Row}(A) = \text{Span} \left\{ \left( \begin{pmatrix} 2 \end{pmatrix} \right) \right\} \)