Welcome to Math 218D-1!

Introduction to Linear Algebra

What is Linear Algebra?
The study of (systems of) linear equations

Like: \( y = 3x + 2 \quad \rightarrow \quad -3x + y = 2 \)
(usually put variables on the left & constants on the right)

Or: \[
\begin{align*}
\left\{ \begin{array}{c}
x + y + z = 1 \\
y - z = -3
\end{array} \right. \\
\end{align*}
\]
solve both equations at once

\[ \uparrow \quad \uparrow \quad \uparrow \] (arrange in columns to keep things tidy)

Linear means: equations that involve only sums of (number) \cdot (variable) or (number)

Not: \( xy + z = 1 \) \hspace{1cm} \( x + 3 = y^2 \) \hspace{1cm} \( e^x = \cos(y) \)

\( \text{product of variables} \) \hspace{1cm} \( \text{power of a variable} \) \hspace{1cm} \( \text{complicated functions} \)

Why linear algebra?

- It's simple enough to understand very well & program computers to do quickly.
It's powerful enough to solve a huge range of different problems.

Eg:

Here's a map of roads in the town square:

```
120 250 120 70
175 530 390
```

cars/hr

Question: How many cars/hr travel on the unlabeled roads?

Step 0: When you have an unknown quantity, give it a name!

Observation:

# cars entering each intersection = # cars leaving it

**A:** \( 120 + w = 250 + x \)

**B:** \( 120 + x = 70 + y \)

**C:** \( 530 + y = 390 + z \)

**D:** \( 115 + z = 175 + w \)

This is a system of 4 linear equations in 4 unknowns!
Question: You know a priori that there are infinitely many solutions. How?

Question: What must be true about the known quantities for a solution to exist?

Linear algebra is a set of tools for solving equations. It is your job to turn your question into a linear algebra problem (that a computer can solve) and interpret the answer.

Eg: An asteroid has been observed at coordinates: (0,2), (2,1), (1,-1), (-1,-2), (-3,1), (-1,1)

Equation for an ellipse:

\[ x^2 + By^2 + Cxy + Dx + Ey + F = 0 \]

Wait! Isn't this a nonlinear equation? —
For our points to lie on the ellipse, substitute the coordinates into \((x,y)\) \& these should hold:

\[
\begin{align*}
(0,2): & \quad 0 + 4B + 0 + 0 + 2E + F = 0 \\
(2,1): & \quad 4 + B + 2C + 2D + E + F = 0 \\
(1,-1): & \quad 1 + B - C + D - E + F = 0 \\
(-1,-2): & \quad 1 + 4B + 2C - D - 2E + F = 0 \\
(-3,1): & \quad 9 + B - 3C + D - 3E + F = 0 \\
(-1,1): & \quad 1 + B - C - D + E + F = 0
\end{align*}
\]

This is a system of six linear equations in 5 variables.

**Note**
- The variables are the coefficients \(B,C,D,E,F\).
- Remember, we're finding the equation of the ellipse.

**NB:** There is no solution — the points do not lie on an ellipse (perhaps due to measurement error).

**Question:** What is the best approximate solution?

-> "Least squares" (week 8)

**Answer:** [demo]
Historical note: Gauss invented much of what you'll learn to (correctly) predict the orbit of the asteroid Ceres in 1801.

Note on demos: I created these to help give you a geometric understanding of linear algebra.
→ It took a lot of work.
→ Really, it was hard.
→ Why would I do that? I want you to have a geometric understanding.

Upshot: Play with the demos! Don't turn off your brain when we do geometry! You will be expected to draw pictures on exams!

Eg: In a population of rabbits,
(1) Half survive their first year.
(2) Half of those survive their second year.
(3) The maximum life span is 3 years.
(4) Each rabbit produces (on average) 0, \( \frac{6}{8} \), and \( \frac{8}{8} \) offspring in years 0, 1, 2, respectively.

Question: How many rabbits will there be in 100 years?
Step 0: Give names to the unknowns

\[
X_n: \text{# rabbits aged 0 in year } n
\]
\[
Y_n: \text{# rabbits aged 1 in year } n
\]
\[
Z_n: \text{# rabbits aged 2 in year } n
\]

Rules:
\[
X_{2021} = 6Y_{2020} + 8Z_{2020}
\]
\[
Y_{2021} = \frac{1}{2}X_{2020}
\]
\[
Z_{2021} = \frac{1}{2}Y_{2020}
\]

A system of equations of this form is called a difference equation. We'll solve them using eigenvalues \& diagonalization (week 10).

[Demo] It looks like eventually,

- The population doubles each year
- The ratio of rabbits aged 0:1:2 is ≈ 16:4:1

Comes from: \( \langle 1 \rangle \) is an eigenvector of \[
\begin{bmatrix}
0 & 6 & 8 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] w/eigenvalue 2.

Other examples:
- Google PageRank lets you search the Web with a Markov chain - a special type of difference equation.
• Netflix knows what movies you'll like using the Singular Value Decomposition (weeks 13-14).

Geometry of Solutions

Convention: given a system of linear equations, put the constant term on the right of the =, and put the variables on the left organized in columns.

\[
\begin{align*}
120 + w &= 250 + x & -x & + w &= 130 \\
120 + x &= 70 + y & \rightarrow x - y &= -50 \\
570 + y &= 390 + z & \rightarrow y - z &= -140 \\
115 + z &= 175 + w & \rightarrow z - w &= 60
\end{align*}
\]

Def: The solution set of a system of equations is the set of all values for the variables making all equations true simultaneously.

Question: What does the solution set of a system of linear equations look like?
One equation in 2 variables:
\[ x - 2y = 1 \implies y = \frac{1}{2}x - \frac{1}{2} \]

One equation in 3 variables:
\[ x + ty + z = 1 \implies z = 1 - x - y \]

One equation in 4 variables: "3-plane in 4-space"

Note on dimensions: Students often want to say "the fourth dimension is time." Einstein used \( \mathbb{R}^4 \) (4-space) to model spacetime, but it models lots of other things too. (like traffic around the town square...)

2 equations in 2 variables:
\[ \begin{align*}
   x - 2y &= 1 \\
   3x + 2y &= 11
\end{align*} \]

Where are both true?

Intersection of 2 lines.
(Answer: \((3,1)\))
What else can happen?

\[ \begin{align*}
  x - 2y &= 1 \\
  3x - 6y &= 3
\end{align*} \]

Same line: \( \infty \) solutions.

\[ \begin{align*}
  x - 2y &= 1 \\
  3x - 6y &= 6
\end{align*} \]

Parallel lines: 0 solutions

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2 equations in 3 variables:

\[ \begin{align*}
  x + y + z &= 1 \\
  x - z &= 0
\end{align*} \]

Intersection of two planes in space [duh]

In this case, it's a line.

3 equations in 3 variables:

\[ \begin{align*}
  x + y + z &= 1 \\
  x - z &= 0 \\
  y &= 0
\end{align*} \]

Intersection of three planes in space: in this case it's a point.
Question: How many "ways" can 3 planes in space intersect?

Answer: 8

Syllabus Stuff: see the syllabus for details.

- Course materials, calendar, resources, links, etc. are on the course webpage:
  https://services.math.duke.edu/~jldr/2223f-218/

- We will use Sakai for:
  - Announcements
  - Gradebook
  - Gradescope
  
  !! Sakai is now better integrated with Gradescope. Please use the Gradescope tab on Sakai instead of going to gradescope.com.

- Ed Discussion for asking questions (replaces Piazza).
  
  !! Don't email us w/math questions! Post it here instead - then everyone sees it & anyone can answer.
Textbook:
• Strang, "Introduction to Linear Algebra," 5th ed. We’ll only follow this loosely. Also see
• Margalit – Rabinoff, "Interactive Linear Algebra" (on the course website)

Quizzes: a 10-minute small-group quiz will be held at the beginning of each discussion section. It’s very basic – just tests if you’ve looked over your notes.

Homework: due Wednesday 11:59 pm every week.
• Meant to be long and hard: you need practice to learn math, and practice takes time.
• Scan & submit on Gradescope. Use a scanning app!
• Tag the pages on Gradescope with the problems on that page!

Midterms: 3 of them, during discussion slots.
Final: as scheduled by the registrar.
Help! • Come to office hours!
  • Ask on Ed Discussion
  • See course webpage.

Recorded Lecture:
Basics of vector & matrix algebra.
Watch before Thursday. (on Warp Wine)
HW#1 also covers that material.