Math 218D-1: Homework #9
Answer Key

1. Compute
   \[ \det \left[ \begin{array}{ccc} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{array} \right] - \lambda I_3 \]
   where \( \lambda \) is an unknown real number. Your answer will be a function of \( \lambda \).

   \textbf{Solution.}
   \[-x^3 + 4x^2 + 12x + 45\]

2. a) Compute the determinants of the matrices in HW8#14 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!

   b) Compute the determinants of the matrices in HW8#14(b) and (d) again using Sarrus’ scheme.

   c) For the matrix of HW8#14(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus’ scheme. Did you get the determinant?

   \textbf{Solution.}
   c) If you try to apply Sarrus’ scheme to this matrix, you get 1140, which is not equal to its determinant.

3. Consider the matrix
   \[ A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}. \]

   a) Compute the cofactor matrix \( C \) of \( A \).

   b) Compute \( AC^T \). What is the relationship between \( C^T \) and \( A^{-1} \)?

   \textbf{Solution.}
   a) \( C = \begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \)

   b) \( AC^T = 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

   \( A^{-1} = \frac{1}{\det(A)} C^T \)

4. Consider the \( n \times n \) matrix \( F_n \) with 1’s on the diagonal, 1’s in the entries immediately below the diagonal, and \(-1\)'s in the entries immediately above the diagonal:

   \[ F_2 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \ldots \]
a) Show that \( \det(F_2) = 2 \) and \( \det(F_3) = 3 \).

b) Expand in cofactors to show that \( \det(F_n) = \det(F_{n-1}) + \det(F_{n-2}) \).

c) Compute \( \det(F_4), \det(F_5), \det(F_6), \det(F_7) \) using b).

This shows that \( \det(F_n) \) is the \( n \)th Fibonacci number. (The sequence usually starts with 1, 1, 2, 3, \ldots, so our \( \det(F_n) \) is the usual \( n + 1 \)st Fibonacci number.)

Solution.

b) First notice that

\[
F_n = \begin{pmatrix}
1 & -1 & 0 & \cdots \\
1 & 1 & -1 & 0 & \cdots \\
0 & 1 & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & & & F_{n-2}
\end{pmatrix}
\]

Expanding cofactors along the first column gives

\[
\det(F_n) = \det(F_{n-1}) - \det\begin{pmatrix}
-1 & 0 & 0 & \cdots \\
1 & 1 & 0 & \cdots \\
0 & 1 & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & & & F_{n-2}
\end{pmatrix}
\]

Expanding cofactors of this last matrix along the first row gives \( \det(F_n) = \det(F_{n-1}) + \det(F_{n-2}) \).

c) \( \det(F_4) = 2 + 3 = 5; \det(F_5) = 3 + 5 = 8; \det(F_6) = 5 + 8 = 13; \det(F_7) = 8 + 13 = 21 \).

5. Let \( A \) be an \( n \times n \) invertible matrix with integer (whole number) entries.

a) Explain why \( \det(A) \) is an integer.

b) If \( \det(A) = \pm 1 \), show that \( A^{-1} \) has integer entries.

c) If \( A^{-1} \) has integer entries, show that \( \det(A) = \pm 1 \).

Solution.

a) If you compute the determinant by expanding cofactors, then you will be adding products of integers together.

b) By a), the cofactor matrix also has integer entries, so \( A^{-1} = \pm C^T \) has integer entries.

c) We have \( 1 = \det(A) \det(A^{-1}) \); if both determinants are integers, then they must be \( \pm 1 \).

6. Let \( V \) be a subspace of \( \mathbb{R}^n \). The matrix for reflection over \( V \) is

\[
R_V = I_n - 2P_{V^\perp},
\]
where \( P_{V\perp} = I_n - P_V \) is the projection matrix onto \( V^\perp \).

**a)** Suppose that \( V \) is the line in the picture. Draw the vectors \( R_V x_1, R_V x_2, R_V x_3, \) and \( R_V x_4 \) as points in the plane.

**b)** Show that any reflection matrix \( R_V \) is orthogonal.

[**Hint:** Recall that \( P^2_{V\perp} = P_{V\perp} = P_{V\perp}^T \).]

**c)** Let \( V \) be the plane \( x + y + z = 0 \). Compute \( R_V \) and \( \det(R_V) \).

**d)** Let \( V \) be any plane in \( \mathbb{R}^3 \). Prove that \( \det(R_V) = -1 \), as follows: choose an orthonormal basis \( \{u_1, u_2\} \) for \( V \), and let \( u_3 = u_1 \times u_2 \). Show that the matrix \( A \) with columns \( u_1, u_2, u_3 \) has determinant 1, and that \( R_V A \) has determinant \(-1\).

Summary: a reflection over a plane in \( \mathbb{R}^3 \) has determinant \(-1\).

**e)** Now compute \( \det(R_L) \), where \( L \) is the x-axis in \( \mathbb{R}^3 \).

**Solution.**

**b)** We have

\[
R_V^T R_V = (I_3 - 2P_{V\perp}^T)(I_3 - 2P_{V\perp}) = I_3 - 2P_{V\perp}^T - 2P_{V\perp} + 4P_{V\perp}^T P_{V\perp} = I_3
\]

since \( P_{V\perp}^T = P_{V\perp} \) and \( P_{V\perp}^2 = P_{V\perp} \).
c) \( R_V = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} \); \( \det(R_V) = -1 \)

d) The determinant of \( A \) is \((u_1 \times u_2) \cdot u_3 = u_3 \cdot u_3 = 1\). We have
\[
R_V u_1 = u_1 \quad R_V u_2 = u_2 \quad R_V u_3 = -u_3
\]
since \( u_1, u_2 \in L^\perp \) and \( u_3 \in L \). Hence \( R_V A \) has columns \( u_1, u_2, -u_3 \), so \( \det(R_V) = \det(R_V A) = -1 \).

e) This determinant is equal to +1.

7. Use a cross product to find an implicit equation for the plane

\[ V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}. \]

Compare HW6#6(a).

**Solution.**

We can compute \( V^\perp \) using a cross product:
\[
V^\perp = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} \right\}.
\]

Hence \( V = \{(x, y, z) \in \mathbb{R}^3 : -3x + 6y - 3z = 0\} \).

8. a) Let \( v = (a, b) \) and \( w = (c, d) \) be vectors in the plane, and let \( A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \). By taking the cross product of \((a, b, 0)\) and \((c, d, 0)\), explain how the right-hand rule determines the sign of \( \det(A) \).

b) Using the identity
\[
\left[ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},
\]

explain how the right-hand rule determines the sign of a \(3 \times 3\) determinant.

**Solution.**

a) We have
\[
\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \times \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ad - bc \end{pmatrix}.
\]

This points up if \( \det(A) > 0 \) and down otherwise. By the right-hand rule, \( \det(A) > 0 \) if and only if \( u \) is counterclockwise from \( v \).
b) The dot product in \((u \times v) \cdot w\) is positive if and only if \(w\) makes an acute angle with \(u \times v\). This means that, if you apply the right-hand rule to \(u, v\), then \(w\) points in the general direction of your thumb.

9. Decide if each statement is true or false, and explain why.
   a) The determinant of the cofactor matrix of \(A\) equals the determinant of \(A\).
   b) \(u \times v = v \times u\).
   c) If \(u \times v = 0\) then \(u \perp v\).

Solution.

a) False.

b) False: \(u \times v = -v \times u\).

c) False.

10. For each matrix \(A\) and each vector \(v\), decide if \(v\) is an eigenvector of \(A\), and if so, find the eigenvalue \(\lambda\).

   a) \(\begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}\)
   b) \(\begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\)
   c) \(\begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}\)
   d) \(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\)
   e) \(\begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\)

Solution.

a) not an eigenvector

b) yes, \(\lambda = 5\)

c) yes, \(\lambda = -3\)

d) yes, \(\lambda = 0\)

e) not an eigenvector
11. For each matrix $A$ and each number $\lambda$, decide if $\lambda$ is an eigenvalue of $A$; if so, find a basis for the $\lambda$-eigenspace of $A$.

a) \[
\begin{pmatrix}
-5 & -14 \\
3 & 8
\end{pmatrix}
\], $\lambda = 1$

b) \[
\begin{pmatrix}
-5 & -14 \\
3 & 8
\end{pmatrix}
\], $\lambda = -1$

c) \[
\begin{pmatrix}
2 & 3 & -15 \\
5 & -7 & 31 \\
2 & -3 & 13
\end{pmatrix}
\], $\lambda = 3$

d) \[
\begin{pmatrix}
2 & 3 & -15 \\
5 & -7 & 31 \\
2 & -3 & 13
\end{pmatrix}
\], $\lambda = 2$

e) \[
\begin{pmatrix}
3 & 1 & -2 \\
-2 & 0 & 4 \\
-1 & -1 & 4
\end{pmatrix}
\], $\lambda = 2$

f) \[
\begin{pmatrix}
1 & 1 & -2 \\
-2 & -2 & 4 \\
-1 & -1 & 2
\end{pmatrix}
\], $\lambda = 0$

g) \[
\begin{pmatrix}
7 & 0 & 0 \\
0 & 7 & 0 \\
0 & 0 & 7
\end{pmatrix}
\], $\lambda = 7$

h) \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\], $\lambda = 0$

Solution.

a) \[\left\{ \begin{pmatrix} 7 \\ -3 \end{pmatrix} \right\} \]

b) not an eigenvalue

c) not an eigenvalue

d) \[\left\{ \begin{pmatrix} -6 \\ 7 \\ 3 \end{pmatrix} \right\} \]

e) \[\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \]

f) \[\left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} , \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\} \]

g) \[\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

h) \[\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \]

12. Suppose that $A$ is an $n \times n$ matrix such that $Av = 2v$ for some $v \neq 0$. Let $C$ be any invertible matrix. Consider the matrices

a) $A^{-1}$

b) $A + 2I_n$

c) $A^3$

d) $CAC^{-1}$.

Show that $v$ is an eigenvector of a)–c) and that $Cv$ is as eigenvector of d), and find the eigenvalues.

Solution.

a) $Av = 2v \implies A^{-1}(2v) = v \implies A^{-1}v = \frac{1}{2}v$.

b) $(A + 2I_n)v = Av + 2v = 2v + 2v = 4v$.

c) $A^3v = A^2(Av) = 2A^2v = 2A(2v) = 4Av = 8v$.

d) $(CAC^{-1})v = CAv = C(2v) = 2Cv$.

13. Here is a handy trick for computing eigenvectors of a $2 \times 2$ matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ matrix with eigenvalue $\lambda$. Explain why $\begin{pmatrix} -b \\ a - \lambda \end{pmatrix}$ and $\begin{pmatrix} d - \lambda \\ c \end{pmatrix}$ are $\lambda$-eigenvectors of $A$ if they are nonzero.

For which matrices $A$ does this trick fail?
Solution.
Since $\lambda$ is an eigenvalue of $A$, the matrix $A - \lambda I_2$ is not invertible, so its rows are multiples of each other. The first row has zero dot product with $\begin{pmatrix} -b \\ a - \lambda \end{pmatrix}$, and the second has zero dot product with $\begin{pmatrix} d - \lambda \\ -c \end{pmatrix}$.

This will fail when both $\begin{pmatrix} -b \\ a - \lambda \end{pmatrix}$ and $\begin{pmatrix} d - \lambda \\ -c \end{pmatrix}$ are zero, which happens when $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I_2$, in which case any nonzero vector is a $\lambda$-eigenvector.

14. a) Show that $A$ and $A^T$ have the same eigenvalues.

b) Give an example of a $2 \times 2$ matrix $A$ such that $A$ and $A^T$ do not share any eigenvectors.

c) A stochastic matrix is a matrix with nonnegative entries such that the entries in each column sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix. [Hint: show that $(1, 1, \ldots, 1)$ is an eigenvector of $A^T$.]

Solution.

a) They have the same characteristic polynomial:
$$\det(A^T - \lambda I_n) = \det((A - \lambda I_n)^T) = \det(A - \lambda I_n).$$

b) There are many answers. For instance, the matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ has eigenspaces spanned by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \end{pmatrix}$, but $A^T = \begin{pmatrix} 1 & -2 \\ 4 & 1 \end{pmatrix}$ has eigenspaces spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

c) The condition on the columns means
$$A^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$  

Hence $(1, 1, \ldots, 1)$ is an eigenvector of $A^T$ with eigenvalue 1.

15. a) Find all eigenvalues of the matrix
$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 & -5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$  

b) Explain how to find the eigenvalues of any triangular matrix.

Solution.

a) The matrix $A - \lambda I_5$ is upper-triangular, so its determinant is the product of its diagonal entries:
$$p(\lambda) = \det(A - \lambda I_5) = (1 - \lambda)(3 - \lambda)(1 - \lambda)(2 - \lambda)(-1 - \lambda).$$
Hence the eigenvalues are just the diagonal entries: 1, 3, 1, 2, −1.

b) The above argument works for any triangular matrix.

16. Recall that an orthogonal matrix is a square matrix with orthonormal columns. Prove that any (real) eigenvalue of an orthogonal matrix $Q$ is $±1$.

Solution.
We have $\|Qx\| = \|x\|$ for all $x$, so if $Qx = \lambda x$ then $|\lambda| = 1$.

17. Suppose that $A$ is a square matrix such that $A^k$ is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of $A$.

Solution.
If $Av = \lambda v$ then $0 = A^k v = \lambda^k v$, so $\lambda^k = 0$ and hence $\lambda = 0$.

18. Decide if each statement is true or false, and explain why.
   a) If $v, w$ are eigenvectors of a matrix $A$, then so is $v + w$.
   b) An eigenvalue of $A + B$ is the sum of an eigenvalue of $A$ and an eigenvalue of $B$.
   c) An eigenvalue of $AB$ is the product of an eigenvalue of $A$ and an eigenvalue of $B$.
   d) If $Ax = \lambda x$ for some vector $x$, then $\lambda$ is an eigenvalue of $A$.
   e) A matrix with eigenvalue 0 is not invertible.
   f) The eigenvalues of $A$ are equal to the eigenvalues of a row echelon form of $A$.

Solution.
   a) False: they need to have the same eigenvalue for this to be true.
   b) False.
   c) False.
   d) False: the vector needs to be nonzero.
   e) True.
   f) False.