Math 218D-1: Homework #8

due Wednesday, October 26, at 11:59pm

1. **a)** Let $v, w \in \mathbf{R}^n$. Show that

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

if $v \perp w$.

b) Let V be a subspace of $\mathbf{R}^n$, let $b \in \mathbf{R}^n$, and let $v \in V$. Use **a)** and the fact that $b - b_V \in V^\perp$ to show that

$$\|b - v\|^2 = \|b - b_V\|^2 + \|b_V - v\|^2.$$

Use this to prove that b_V really is the closest vector in V to b.

c) Let V be a subspace of $\mathbf{R}^n$ and let $b \in \mathbf{R}^n$. Use **a)** to show that $\|b_V\| \leq \|b\|$, with equality if and only if $b \in V$.

2. For each set of vectors, decide if they are orthogonal, orthonormal, or neither; then compute $Q^T Q$, where Q is the matrix with the vectors as columns.

a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ **b)** $\left\{ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

c) $\left\{ \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$ **d)** $\left\{ \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{18}}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \right\}$

3. The following subspaces V are given as the span of an *orthogonal* set of vectors. For each subspace V and vector b, compute the orthogonal projection b_V using the *projection formula*, and compute the projection matrix P_V using the *outer product formula*.

a) $V = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$ $\qquad b = \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$

b) $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ $\qquad b = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$

c) $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ $\qquad b = \begin{pmatrix} 4 \\ 2 \\ -4 \\ 2 \end{pmatrix}$

d) $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ $\qquad b = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$

e) $V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} \right\}$ $\qquad b = \begin{pmatrix} 9 \\ -2 \\ 3 \end{pmatrix}$

4. For each subspace V of Problem 3, scale the spanning vectors to find an *orthonormal* basis of V, and (re)compute the projection matrix P_V using the formula $P_V = QQ^T$. (Your answers should be exact, in terms of square roots.)

5. Suppose that $\{u_1, u_2, \ldots, u_n\}$ is an orthonormal basis of $\mathbf{R}^n$. Use the outer product formula to explain why

$$I_n = u_1u_1^T + u_2u_2^T + \cdots + u_nu_n^T.$$

6. Give an example of each of the following, or explain why no such example exists.

a) A matrix Q with orthonormal columns, but $QQ^T \neq I_n$.

b) Two nonzero orthogonal vectors that are linearly dependent.

c) An orthonormal basis for the plane $x + y + z = 0$.

7. Use the Gram–Schmidt process to find orthogonal bases of the following subspaces.

a) $\text{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}1\\-1\\2\\-2\end{pmatrix}, \begin{pmatrix}1\\-3\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\4\\-1\end{pmatrix}\right\}$

c) $\text{Span}\left\{\begin{pmatrix}1\\-2\\3\end{pmatrix}, \begin{pmatrix}2\\-5\\1\end{pmatrix}, \begin{pmatrix}-1\\5\\12\end{pmatrix}\right\}$ **d)** $\text{Nul}\begin{pmatrix}1 & 2 & -1 & 4\\3 & 6 & -3 & 12\end{pmatrix}$

8. Consider the subspace $V = \text{Col}(A)$, where

$$A = \begin{pmatrix}1 & 1 & 2\\-1 & -3 & 2\\2 & 2 & 4\\-2 & -1 & -1\end{pmatrix}.$$

Find an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ of $\mathbf{R}^4$ such that $\{u_1, u_2, u_3\}$ is a basis for V. Your answer should be exact, in terms of square roots.

9. Consider the plane

$$V = \text{Span}\left\{\begin{pmatrix}1\\-1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\1\end{pmatrix}\right\}$$

and the vector

$$v = \begin{pmatrix}4\\-1\\3\\3\end{pmatrix} = 2\begin{pmatrix}1\\-1\\2\\1\end{pmatrix} + \begin{pmatrix}2\\1\\-1\\1\end{pmatrix} \in V.$$

Find all vectors **contained in** V that are orthogonal to v.

10. For each of the following matrices A and vectors b, find the QR decomposition of A, and find the least-squares solution of $Ax = b$ by back-substitution in $R\widehat{x} = Q^T b$. Your answers should be exact, in terms of square roots.

a) $A = \begin{pmatrix}1 & 1\\1 & 0\\0 & 2\end{pmatrix}, \; b = \begin{pmatrix}1\\1\\1\end{pmatrix}$ **b)** $A = \begin{pmatrix}1 & 1 & 2\\-1 & -3 & 2\\2 & 2 & 4\\-2 & -1 & -1\end{pmatrix}, \; b = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$

11. In this problem, we use a *QR* decomposition to *quickly* compute the best-fit parabola with specified y-values at $x = -2, -1, 1, 2$, as in HW7#3.

a) Compute the matrix A such that the least squares solution of $A(C, D, E) = (b_1, b_2, b_3, b_4)$ gives the coefficients of the parabola $y = Cx^2 + Dx + E$ that best fits the data points $(-2, b_1)$, $(-1, b_2)$, $(1, b_3)$, $(2, b_4)$. (Presumably you computed this in HW7#3.)

b) Find the QR decomposition of A.

c) Find the best-fit parabola through the points $(-2, 3)$, $(-1, -1)$, $(1, 1)$, $(2, 3)$ by back-substitution in $R\widehat{x} = Q^T b$. You should get the same answer as in HW7#3.

Note that we can now repeat part **c)** with new y-values in $O(n^2)$ time.

12. Recall that an *orthogonal* matrix is a square matrix with ortho*normal* columns.

a) If Q is an orthogonal matrix, show that Q^{-1} is orthogonal.

b) If Q_1 and Q_2 are orthogonal matrices of the same size, show that $Q_1 Q_2$ is orthogonal.

c) If Q is orthogonal, show that $\det(Q) = \pm 1$.

13. Decide if each statement is true or false, and explain why.

a) A matrix with orthogonal columns has full row rank.

b) If $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors, then it is orthogonal.

c) If $\{v_1, v_2\}$ is a basis for a plane V, then for any vector b,

$$b_V = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} v_2.$$

d) If Q has orthonormal columns, then the distance from x to y equals the distance from Qx to Qy.

e) If $A = QR$ is a QR-factorization of a matrix A, then the rows of Q form an orthonormal basis for Row(A).

14. Compute the determinants of the following matrices *using Gaussian elimination*.

$$\textbf{a)}\ \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$$

$$\textbf{c)}\ \begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} \qquad \textbf{d)}\ \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

15. Suppose that

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10.$$

Find the determinants of the following matrices.

a) $\begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$ **b)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix}$ **c)** $\begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$

d) $\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ **e)** $\begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix}$ **f)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

g) $2\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **h)** $\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **i)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$

j) $-\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ **k)** $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{3}$ **l)** $\begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$

16. Find $\det(E)$ when:

a) E is the elementary matrix for a row replacement.

b) E is the elementary matrix for $R_i \times= c$.

c) E is the elementary matrix for a row swap.

17. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?.

18. Let V be a subspace of $\mathbf{R}^n$ and let P_V be the projection matrix onto V.

a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.

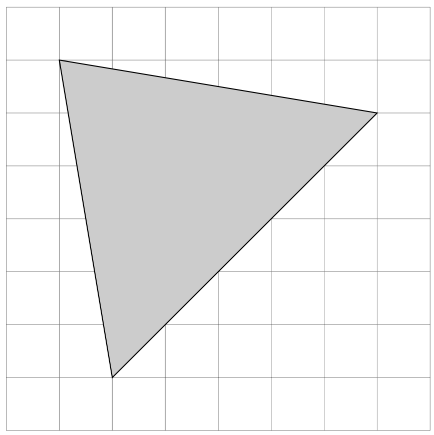
b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

19. Let A be an $n \times n$ matrix with columns $v_1, v_2, \ldots, v_n$.

a) Show that if $\{v_1, v_2, \ldots, v_n\}$ is orthogonal then $|\det(A)| = \|v_1\|\,\|v_2\| \cdots \|v_n\|$.
[**Hint:** Compute $A^T A$ and its determinant.]

b) Suppose that A is invertible. Show that $|\det(A)| \le \|v_1\|\,\|v_2\| \cdots \|v_n\|$, with equality if and only if the set $\{v_1, v_2, \ldots, v_n\}$ is orthogonal.
[**Hint:** Use Problem 1(c) and the QR decomposition of A.]

20. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



21. Decide if each statement is true or false, and explain why.

a) $\det(A+B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \dfrac{\det(A)\det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3\det(A)$.

e) If A^5 is invertible then A is invertible.

f) The determinant of A is the product of its diagonal entries.

g) If the columns of A are linearly dependent, then $\det(A) = 0$.

h) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.