Math 218D-1: Homework #6
Answer Key

1. For each pair of vectors $v$ and $w$, draw $\text{Span}\{v\}$, and compute and draw the projection $p$ of $w$ onto $\text{Span}\{v\}$.

   a) $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $w = \begin{pmatrix} \cos(123^\circ) \\ \sin(123^\circ) \end{pmatrix}$
   b) $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution.

![Diagram a)](image)

![Diagram b)](image)

2. For each subspace $V$ and vector $b$, compute the orthogonal projection $b_V$ of $b$ onto $V$ by solving a normal equation $A^T Ax = A^T b$, and find the distance from $b$ to $V$.

   a) $V = \text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$
   b) $V = \text{Col} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$
   c) $V = \text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$, $b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$

Solution.

a) $b_V = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$, $\|b - p\| = 3$

b) $b_V = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 5 \end{pmatrix}$, $\|b - p\| = 4$
For each subspace \( V \), compute the orthogonal decomposition \( b = b_V + b_{V^\perp} \) of the vector \( b = (1, 2, -1) \) with respect to \( V \).

3. For each subspace \( V \), compute the orthogonal decomposition \( b = b_V + b_{V^\perp} \) of the vector \( b = (1, 2, -1) \) with respect to \( V \).

   a) \( V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} \)

   b) \( V = \text{Nul} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \)

   c) \( V = \mathbb{R}^3 \)

   d) \( V = \{0\} \)

[Hint: Only part a) requires any work.]

Solution.

a) \( \begin{align*}
    b_V &= \begin{pmatrix} -1/5 \\ 2 \\ -2/5 \end{pmatrix} \\
    b_{V^\perp} &= \begin{pmatrix} 6/5 \\ 0 \\ -3/5 \end{pmatrix}
\end{align*} \)

b) \( \begin{align*}
    b_V &= \begin{pmatrix} 6/5 \\ 0 \\ -3/5 \end{pmatrix} \\
    b_{V^\perp} &= \begin{pmatrix} -1/5 \\ 2 \\ -2/5 \end{pmatrix}
\end{align*} \)

c) \( b_V = b, \quad b_{V^\perp} = 0 \)

d) \( b_V = 0, \quad b_{V^\perp} = b \)

4. Compute the orthogonal decomposition \((3, 1, 3) = b_V + b_{V^\perp}\) with respect to each subspace of \( V \) of HW5#18(a)–(e).

[Hint: Only parts a) and c) require any work, and even c) doesn’t require work if you’re clever enough. In fact, you can solve all five parts by computing two dot products.]

Solution.

Here’s an explanation of the hint. You found in HW5#18(b) that the null space of the matrix

\[
A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}
\]

is the line through \( v = (1, -2, 1) \). Hence you can compute the projection onto \( \text{Nul}(A) \) by \((b \cdot v / v \cdot v) v\). This gives you the decompositions with respect to \( \text{Nul}(A) \) and \( \text{Row}(A) \). If you were clever in HW5#18, you realized that \( \text{Nul}(A^T) = \text{Nul}(A) \) and \( \text{Col}(A) = \text{Row}(A) \), which gives the rest.

a) \( \begin{align*}
    b_V &= \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
    b_{V^\perp} &= \frac{1}{3} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix}
\end{align*} \)
b) \[ b_V = \frac{1}{3} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \quad b_{V \perp} = \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

c) \[ b_V = \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad b_{V \perp} = \frac{1}{3} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \]

d) \[ b_V = \frac{1}{3} \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} \quad b_{V \perp} = \frac{7}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \]

e) \[ b_V = b \quad b_{V \perp} = 0 \]

5. Let \( A \) be an \( m \times n \) matrix, and let \( b \in \mathbb{R}^n \) be a vector. Suppose that \( A^T b = 0 \). Compute the orthogonal decomposition \( b = b_V + b_{V \perp} \) with respect to \( V = \text{Col}(A) \).

**Solution.**
If \( A^T b = 0 \) then \( b \) is in the left null space of \( A \), which is the orthogonal complement of the column space. Thus \( b_V = 0 \) and \( b_{V \perp} = b \).

6. a) Find an implicit equation for the plane \( \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\} \).

[Hint: use HW5#18(a).]

b) Find implicit equations for the line \( \{ (t, -t, t) : t \in \mathbb{R} \} \).
[Hint: use HW5#19(g).]

**Solution.**
\[ a) \ x_1 - 2x_2 + x_3 = 0 \quad b) \begin{cases} x_1 + x_2 = 0 \\ -x_1 + x_3 = 0 \end{cases} \]

7. Show that \( A^T A = 0 \) is only possible when \( A = 0 \).

**Solution.**
The diagonal entries of \( A^T A \) are the dot products of each column with itself.

8. Let \( Q \) be an \( n \times n \) matrix such that \( Q^T Q = I_n \) (so \( Q^T = Q^{-1} \)).

a) Show that the columns of \( Q \) are unit vectors.

b) Show that the columns of \( Q \) are orthogonal to each other.

c) Show that the rows of \( Q \) are also orthogonal unit vectors.

d) Find all \( 2 \times 2 \) matrices \( Q \) such that \( Q^T Q = I_2 \).
Such a matrix $Q$ is called orthogonal.\footnote{I am not responsible for this terminology.}

**Solution.**

a) The $(i, i)$ entry of $Q^TQ$ is the dot product of the $i$th column of $Q$ with itself.

b) The $(i, j)$ entry of $Q^TQ$ is the dot product of the $i$th column of $Q$ with the $j$th column.

c) Since $Q^T = Q^{-1}$ we also have $QQ^T = I_n$, which means parts (a) and (b) also hold for the rows.

d) $Q = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ or $Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}$.

9. Construct a $3 \times 3$ matrix $A$, with no zero entries, whose columns are orthogonal to each other. Compute $A^TA$, and explain why this matrix is diagonal.

**Solution.**

We have to construct three orthogonal vectors in $\mathbb{R}^3$. There are many answers; one is

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}.$$ 

In this case we have

$$A^TA = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$ 

This is the matrix of dot products of columns of $A$; since different columns are orthogonal, it is a diagonal matrix.

10. Explain why $A$ has full column rank if and only if $A^TA$ is invertible.

**Solution.**

We use the fact that $\text{Nul}(A^TA) = \text{Nul}(A)$. We know that $A$ (resp. $A^TA$) has full column rank if and only if $\text{Nul}(A) = \{0\} = \text{Nul}(A^TA)$, and a square matrix is invertible if and only if it has full column rank.

11. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal complements.

b) If $A$ is a $3 \times 4$ matrix, then $\text{Col}(A)\perp$ is a subspace of $\mathbb{R}^4$.

c) If $A$ is any matrix, then $\text{Nul}(A) = \text{Nul}(A^TA)$.

d) If $A$ is any matrix, then $\text{Row}(A) = \text{Row}(A^TA)$.

e) If every vector in a subspace $V$ is orthogonal to every vector in another subspace $W$, then $V = W^\perp$. 
f) If \( x \) is in \( V \) and \( V^\perp \), then \( x = 0 \).

g) If \( x \) is in a subspace \( V \), then the orthogonal projection of \( x \) onto \( V \) is \( x \).

h) If \( x \) is in the orthogonal complement of a subspace \( V \), then the orthogonal projection of \( x \) onto \( V \) is \( x \).

**Solution.**

a) False.

b) False.

c) True.

d) True.

e) False.

f) True.

g) True.

h) False: it is 0.

12. For each column space \( V \), compute the projection matrix \( P_V \). Verify that \( P_V^2 = P_V \) and that \( P_V^T = P_V \).

\[
a) \ V = \text{Col} \left( \begin{array}{cc}
1 & 1 \\
1 & 0 \\
0 & 2
\end{array} \right) \\
b) \ V = \text{Col} \left( \begin{array}{ccc}
1 & 2 & 1 \\
-1 & 1 & 0 \\
2 & 2 & -1 \\
4 & 3 & 0
\end{array} \right) \\
c) \ V = \text{Col} \left( \begin{array}{ccc}
2 & 2 & -1 \\
-4 & -5 & 5 \\
6 & 1 & 12
\end{array} \right)
\]

**Solution.**

\[
a) \frac{1}{9} \begin{pmatrix}
5 & 4 & 2 \\
4 & 5 & -2 \\
2 & -2 & 8
\end{pmatrix} \\
b) \frac{1}{4} \begin{pmatrix}
3 & 1 & -1 & 1 \\
1 & 3 & 1 & -1 \\
-1 & 1 & 3 & 1 \\
1 & -1 & 1 & 3
\end{pmatrix} \\
c) \frac{1}{195} \begin{pmatrix}
26 & -65 & 13 \\
-65 & 170 & 5 \\
13 & 5 & 194
\end{pmatrix}
\]

13. For each subspace \( V \), compute the projection matrix \( P_V \). Verify that \( P_V^2 = P_V \) and that \( P_V^T = P_V \).

\[
a) \ V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \\
b) \ V = \text{Nul} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}
\]

**Solution.**
14. For each vector $v$, compute the projection matrix onto $V = \text{Span}\{v\}$ using the formula $P_V = vv^T/v \cdot v$.

\begin{align*}
a) & \quad v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
b) & \quad v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \\
c) & \quad v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{(in $\mathbb{R}^n$)}
\end{align*}

**Solution.**

\begin{align*}
a) & \quad \frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \\
b) & \quad \frac{1}{26} \begin{pmatrix} 9 & 0 & 12 & -3 \\ 0 & 0 & 0 & 0 \\ 12 & 0 & 16 & -4 \\ -3 & 0 & -4 & 1 \end{pmatrix} \\
c) & \quad \frac{1}{n} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}
\end{align*}

15. a) Compute $P_V$ for $V = \mathbb{R}^n$.

b) Compute $P_V$ for $V = \{0\}$.

**Solution.**

a) $I_n$  \quad b) 0

16. For each subspace $V$, compute the projection matrix $P_V$.

a) $\{(x, y, x) : x, y \in \mathbb{R}\}$.

b) $\{(x, y, z) \in \mathbb{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations \( \begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases} \)

d) $\{x \in \mathbb{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbb{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the $xy$-plane.

g) The line $\{(t, -t, t) : t \in \mathbb{R}\}$.

[Hint: Compare HW4#18 and HW5#19. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]
17. Consider the matrix

\[
A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},
\]

and let \( V = \text{Col}(A) \).

a) Compute \( P_V \) using the formula \( P_V = A(A^T A)^{-1} A^T \).
b) Compute a basis \( \{ v_1, v_2 \} \) for \( V^\perp = \text{Nul}(A^T) \).
c) Let \( B \) be the matrix with columns \( v_1, v_2 \), and compute \( P_{V^\perp} \) using the formula \( B(B^T B)^{-1} B^T \).
d) Verify that your answers to (a) and (c) sum to \( I_4 \).

(Factor out \( ad - bc \) and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you’ve computed \( P_V \), there’s no need to compute \( P_{V^\perp} \) separately. It’s a lot of extra work!

Solution.

a) \( P_V = \frac{1}{107} \begin{pmatrix} 38 & 43 & 22 & 17 \\ 43 & 74 & 8 & -23 \\ 22 & 8 & 24 & 38 \\ 17 & -23 & 38 & 78 \end{pmatrix} \)

b) \( \left\{ \begin{pmatrix} -4/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -7/3 \\ 5/3 \\ 0 \\ 1 \end{pmatrix} \right\} \)

c) \( P_{V^\perp} = \frac{1}{107} \begin{pmatrix} 69 & -43 & -22 & -17 \\ -43 & 33 & -8 & 23 \\ -22 & -8 & 83 & -38 \\ -17 & 23 & -38 & 29 \end{pmatrix} \)

18. Consider the plane \( V \) defined by the equation \( x + 2y - z = 0 \). Compute the matrix \( P_V \) for orthogonal projection onto \( V \) in two ways:
a) Find a basis for \( V \), put your basis vectors into a matrix \( A \), and use the formula \( P_V = A(A^T A)^{-1} A^T \).

b) Compute the matrix for orthogonal projection \( P_{V\perp} \) onto the line \( V^\perp \) using the formula \( vv^T/v \cdot v \), and subtract: \( P_V = I_3 - P_{V\perp} \).

[Hint: It doesn’t take any work to find a basis for \( V^\perp \).]

If \( V \) is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

Solution.

a) We can take

\[
A = \begin{pmatrix}
-2 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix} \quad \Rightarrow \quad P_V = \frac{1}{6} \begin{pmatrix}
5 & -2 & 1 \\
-2 & 2 & 2 \\
1 & 2 & 5
\end{pmatrix}.
\]

b) Taking \( v = (1, 2, -1) \), we have

\[
P_{V\perp} = \frac{1}{6} \begin{pmatrix}
1 & 2 & -1 \\
2 & 4 & -2 \\
-1 & -2 & 1
\end{pmatrix} \quad \Rightarrow \quad P_V = I_3 - P_{V\perp} = \frac{1}{6} \begin{pmatrix}
5 & -2 & 1 \\
-2 & 2 & 2 \\
1 & 2 & 5
\end{pmatrix}.
\]

The second method is much faster, as it does not involve Gauss–Jordan elimination or inversion.

19. Compute the matrices \( P_1, P_2 \) for orthogonal projection onto the lines through \( a_1 = (-1, 2, 2) \) and \( a_2 = (2, 2, -1) \), respectively. Now compute \( P_1 P_2 \), and explain why it is what it is.

Solution.

\[
P_1 = \frac{1}{9} \begin{pmatrix}
1 & -2 & -2 \\
-2 & 4 & 4 \\
-2 & 4 & 4
\end{pmatrix} \quad P_2 = \frac{1}{9} \begin{pmatrix}
4 & 4 & -2 \\
4 & 4 & -2 \\
-2 & -2 & 1
\end{pmatrix} \quad P_1 P_2 = 0
\]

Evaluating \( P_1 P_2 x \) means first orthogonally projecting \( x \) onto the line through \( a_1 \), then orthogonally projecting the result onto the line through \( a_2 \). But \( a_1 \) and \( a_2 \) are orthogonal.

20. Decide if each statement is true or false, and explain why. In each statement, \( V \) is a subspace of \( \mathbb{R}^n \).

a) The rank of \( P_V \) is equal to \( \text{dim}(V) \).

b) \( P_V P_{V\perp} = 0 \).

c) \( P_V + P_{V\perp} = 0 \).

d) \( \text{Col}(P_V) = V \).

e) \( \text{Nul}(P_V) = V \).

f) \( \text{Row}(P_V) = \text{Col}(P_V) \).
g) \( \text{Nul}(P_V) \perp = \text{Col}(P_V) \).

**Solution.**

a) True.

b) True.

c) False.

d) True.

e) False.

f) True.

g) True.