**Math 218D-1: Homework #5**

due Wednesday, October 5, at 11:59pm

**1.** Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that:

(1) $\dim \operatorname{Col}(A) + \dim \operatorname{Nul}(A)$ is the number of columns of $A$.

(2) $\dim \operatorname{Row}(A) + \dim \operatorname{Nul}(A^T)$ is the number of rows of $A$.

(3) $\dim \operatorname{Row}(A) = \dim \operatorname{Col}(A)$.

[**Hint:** Augment with the identity matrix so you only have to do Gauss–Jordan elimination once. Feel free to use the Sage cell on the website!]

**a)** $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

**d)** $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

**2.** Suppose that $A$ is an invertible $4 \times 4$ matrix. Find bases for its four fundamental subspaces.

**3.** Find an example of a matrix with the required properties, or explain why no such matrix exists.

**a)** The column space contains $(1,2,3)$ and $(4,5,6)$, and the row space contains $(1,2)$ and $(2,3)$.

**b)** The column space has basis $\{(1,2,3)\}$, and the null space has basis $\{(3,2,1)\}$.

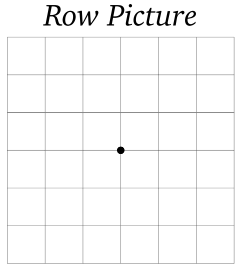
**c)** The dimension of the null space is one greater than the dimension of the left null space.

**d)** A $3 \times 5$ matrix whose row space equals its null space.

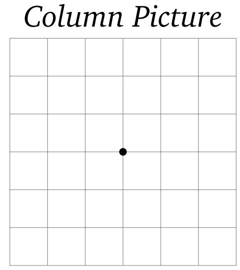
**4.** Draw the four fundamental subspaces of the following matrices, in grids like below.

**a)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

*Row Picture*



*Column Picture*



**5.** For the following matrix $A$, find the pivot positions of $A$ and of $A^T$. Do they have the same pivots? Do they have the same rank?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}$$

**6.** Find a matrix $A$ such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

What is the rank of $A$?

**7.** **a)** If $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$, then $AB =$ __________.

**b)** Find a $2 \times 2$ matrix $A$ such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$. What is the rank of such a matrix? [**Hint:** use HW4#7.]

**8.** **a)** Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$. [**Hint:** Compare HW4#8.]

**b)** Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. [**Hint:** Take transposes.]

**9.** Let $A$ be a $3 \times 3$ matrix of rank 2. Explain why $A^2$ is not the zero matrix.
[**Hint:** Compare Problem 7.]

**10.** **(OPTIONAL AND UNGRADED)** This problem explains why we only consider *square* matrices when we discuss invertibility.

**a)** Show that a tall matrix $A$ (more rows than columns) does not have a right inverse, i.e., there is no matrix $B$ such that $AB = I_m$.

**b)** Show that a wide matrix $A$ (more columns than rows) does not have a left inverse, i.e., there is no matrix $B$ such that $BA = I_n$.

[**Hint:** compare Problem 8.]

**11.** Let $A$ be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "$A$ has full column rank"?

**a)** $\mathrm{Nul}(A) = \{0\}$

**b)** $A$ has rank $m$

**c)** The columns of $A$ are linearly independent

**d)** $\dim \mathrm{Row}(A) = n$

**e)** The columns of $A$ span $\mathbf{R}^m$

**f)** $A^T$ has full column rank

**12.** Let $A$ be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "$A$ has full row rank"?

**a)** $\mathrm{Col}(A) = \mathbf{R}^m$

**b)** $A$ has rank $m$

**c)** The columns of $A$ are linearly independent

**d)** $\dim \mathrm{Nul}(A) = n - m$

**e)** The rows of $A$ span $\mathbf{R}^n$

**f)** $A^T$ has full column rank

**13.** Consider the following vectors:

$$u = \begin{pmatrix} -.6 \\ .8 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

**a)** Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

**b)** Compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

**c)** Find the unit vectors in the directions of $u, v$, and $w$.

**d)** Check the Schwartz inequalities $|u \cdot v| \leq \|u\| \, \|v\|$ and $|v \cdot w| \leq \|v\| \, \|w\|$.

**e)** Find the angles between $u$ and $v$ and between $v$ and $w$.

**f)** Find the distance from $v$ to $w$.

**g)** Find unit vectors $u', v', w'$ orthogonal to $u, v, w$, respectively.

**14.** What is the length of the vector $v = (1, 1, \ldots, 1)$ in $n$ dimensions?

**15.** If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v - w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

**16.** **a)** If $v \cdot w < 0$, what does that say about the angle between $v$ and $w$?

**b)** Find three vectors $u, v, w$ in the $xy$-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$.

**17.** Compute a basis for the orthogonal complement of each of the following spans.

**a)** $\operatorname{Span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$ **b)** $\operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$ **c)** $\operatorname{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}4\\5\\6\end{pmatrix},\begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$

**d)** $\operatorname{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ **e)** $\operatorname{Span}\{\} = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

**f)** $\operatorname{Span}\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$

**18.** Compute a basis for the orthogonal complement of each the following subspaces.

**a)** $\operatorname{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **b)** $\operatorname{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **c)** $\operatorname{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

**d)** $\operatorname{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ **e)** $\operatorname{Span}\left\{\begin{pmatrix}3\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **f)** $\operatorname{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$

[**Hint:** solving **a)**–**d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

**19.** Compute a basis for the orthogonal complement of each the following subspaces.

**a)** $\{(x,y,x) : x,y \in \mathbf{R}\}$.

**b)** $\{(x,y,z) \in \mathbf{R}^3 : x = 2y + z\}$.

**c)** The solution set of the system of equations $\begin{cases} x + \phantom{2}y + z = 0 \\ x - 2y - z = 0. \end{cases}$

**d)** $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix}0&6&8\\\frac{1}{2}&0&0\\0&\frac{1}{2}&0\end{pmatrix}$.

**e)** The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

**f)** The intersection of the plane $x - 2y - z = 0$ with the $xy$-plane.

**g)** The line $\{(t,-t,t) : t \in \mathbf{R}\}$.

[**Hint:** Compare HW4#18.]

**20.** Construct a matrix $A$ with each of the following properties, or explain why no such matrix exists.

**a)** The column space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

**b)** The row space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

**c)** $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 0$.

**d)** A $2 \times 2$ matrix $A$ with no zero entries such that every row of $A$ is orthogonal to every column.

**e)** The sum of the columns of $A$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and the sum of the rows of $A$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.