Math 218D-1: Homework #10

due Wednesday, November 9, at 11:59pm

1. For each 2×2 matrix A, **i)** compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$. Use this to **ii)** find all real eigenvalues, and **iii)** find a basis for each eigenspace, using HW9#13 when applicable. **iv)** Draw and label each eigenspace. **v)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

a) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **d)** $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

2. For each matrix A, **i)** find all real eigenvalues of A, and **ii)** find a basis for each eigenspace. **iii)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

You will probably want to use a computer algebra system to find the roots of the characteristic polynomial. To do so in Sympy, you would type something like:

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print(roots(-x**3 + 13/4*x + 3/2, multiple=True))
# [-1.5, -0.5, 2.0]
```

a) $\begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}$ **b)** $\begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 6 & 2 & 3 \\ -14 & -7 & -12 \\ 1 & 2 & 4 \end{pmatrix}$

Optional (if you want more practice):

d) $\begin{pmatrix} -11 & -54 & 10 \\ -2 & -7 & 2 \\ -21 & -90 & 20 \end{pmatrix}$ **e)** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

f) $\begin{pmatrix} 13 & 18 & -18 \\ -12 & -17 & 18 \\ -4 & -6 & 7 \end{pmatrix}$ **g)** $\begin{pmatrix} -10 & 28 & -18 & -76 \\ -1 & 9 & -6 & -2 \\ 4 & -8 & 7 & 26 \\ 0 & 2 & -2 & 4 \end{pmatrix}$

3. Consider the matrix

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

a) Find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.

b) Find a *different* diagonal matrix D' and a *different* invertible matrix C' such that $A = C'D'C'^{-1}$.

[**Hint:** Try re-ordering the eigenvalues.]

4. Compute the matrix with eigenvalues $0, 1, 2$ and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(There is only one such matrix.)

5. Let A and B be $n \times n$ matrices, and let $v_1, \ldots, v_n$ be a basis of $\mathbf{R}^n$.

a) Suppose that each v_i is an eigenvector of both A and B. Show that $AB = BA$.

b) Suppose that each v_i is an eigenvector of both A and B with the same eigenvalue. Show that $A = B$.

[**Hint:** Hint: use the matrix form of diagonalization.]

6. Let A be an $n \times n$ matrix, and let C be an invertible $n \times n$ matrix. Prove that the characteristic polynomial of CAC^{-1} equals the characteristic polynomial of A.

In particular, A and CAC^{-1} have the same eigenvalues, the same determinant, and the same trace. They are called *similar* matrices.

7. Let V be the plane $x + y + z = 0$, and let $R_V = I_3 - 2P_{V^\perp}$ be the reflection matrix over V, as in HW9#6. Diagonalize R_V without doing any computations.

8. The *Fibonacci numbers* are defined recursively as follows:

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+2} = F_{n+1} + F_n \;\; (n \geq 0).$$

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ In this problem, you will find a closed formula (as opposed to a recursive formula) for the nth Fibonacci number by solving a difference equation.

a) Let $v_n = \binom{F_{n+1}}{F_n}$, so $v_0 = \binom{1}{0}$, $v_1 = \binom{1}{1}$, etc. Find a state change matrix A such that $v_{n+1} = Av_n$ for all $n \geq 0$.

b) Show that the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$, with corresponding eigenvectors $w_1 = \binom{-1}{\lambda_2}$ and $w_2 = \binom{-1}{\lambda_1}$.

[**Hint:** Check that $Aw_i = \lambda_i w_i$ using the relations $\lambda_1\lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$.]

c) Expand v_0 in this eigenbasis: that is, find x_1, x_2 such that $v_0 = x_1 w_1 + x_2 w_2$. (It helps to write x_1, x_2 in terms of λ_1, λ_2.)

d) Multiply $v_0 = x_1 w_1 + x_2 w_2$ by A^n to show that

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

e) Use this formula to explain why F_{n+1}/F_n approaches the golden ratio when n is large.

9. Pretend that there are three Red Box kiosks in Durham. Let x_t, y_t, z_t be the number of copies of Prognosis Negative at each of the three kiosks, respectively, on day t. Suppose in addition that a customer renting a movie from kiosk i will return the movie the next day to kiosk j, with the following probabilities:

Returning to kiosk	Renting from kiosk 1	2	3
1	30%	40%	50%
2	30%	40%	30%
3	40%	20%	20%

For instance, a customer renting from kiosk 3 has a 50% probability of returning it to kiosk 1.

a) Let $v_t = (x_t, y_t, z_t)$. Find the state change matrix A such that $v_{t+1} = Av_t$.

b) Diagonalize A. What are its eigenvalues?

[**Hint:** A is a stochastic matrix, so you know one eigenvalue by HW9#14(c).]

c) If you start with a total of 1 000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk? Does it matter what the initial state is?

This is an example of a stochastic process, and is an important application of eigenvalues and eigenvectors.

10. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find a closed formula for A^n: that is, an expression of the form

$$A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},$$

where $a_{ij}(n)$ is a function of n.

11. Give an example of each of the following, or explain why no such example exists.

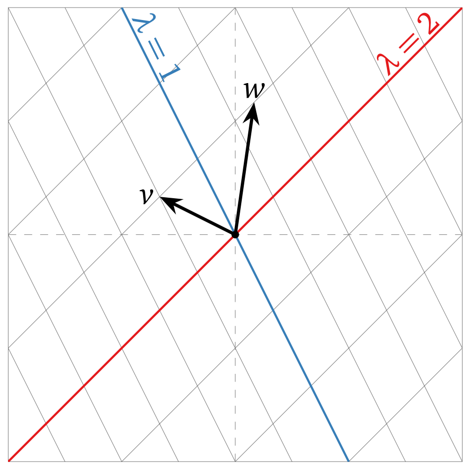
a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.

b) A 2×2 orthogonal matrix with no real eigenvalues.

12. A certain 2×2 matrix A has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

a) Draw Av, A^2v, and Aw.

b) Compute the limit of $A^n v / \|A^n v\|$ as $n \to \infty$.



13. A certain diagonalizable 2×2 matrix A is equal to CDC^{-1}, where C has columns w_1, w_2 pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.

a) Draw $C^{-1}v$ on the left.

b) Draw $DC^{-1}v$ on the left.

c) Draw $Av = CDC^{-1}v$ on the right.

d) What happens to $A^n v$ as $n \to \infty$?