MATH 218D-1
MIDTERM EXAMINATION 3

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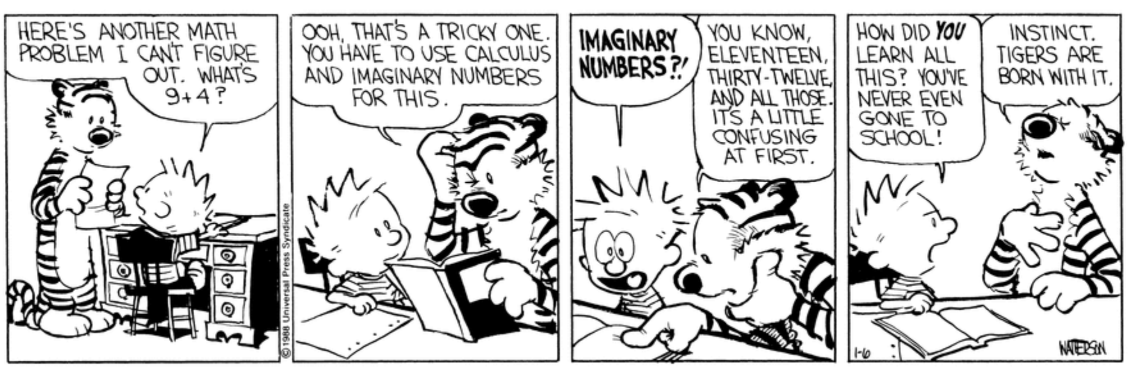
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. You may bring a 3×5**-inch note card** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1. [15 points]

a) For each of the following quadratic forms q_i, find the symmetric matrix S_i such that $q_i(x) = x^T S_i x$.

$$q_1(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + 18x_3^2 - 4x_1x_2 + 12x_1x_3 - 10x_2x_3$$
$$q_2(x_1, x_2, x_3) = x_1^2 + 5x_3^2 + 6x_1x_2 + 18x_1x_3 + 14x_2x_3$$
$$q_3(x_1, x_2, x_3) = 2x_1^2 + 9x_2^2 + 10x_3^2 + 8x_1x_2 - 8x_1x_3 - 14x_2x_3$$

$$S_1 = \begin{pmatrix} 2 & -2 & 6 \\ -2 & 1 & -5 \\ 6 & -5 & 18 \end{pmatrix} \qquad S_2 = \begin{pmatrix} 1 & 3 & 9 \\ 3 & 0 & 7 \\ 9 & 7 & 5 \end{pmatrix} \qquad S_3 = \begin{pmatrix} 2 & 4 & -4 \\ 4 & 9 & -7 \\ -4 & -7 & 10 \end{pmatrix}$$

b) One of the quadratic forms in **a)** is positive-definite. Which is it? q_3

c) Find the LDL^T decomposition of the symmetric matrix associated to the positive-definite quadratic form you identified in **b)**.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 2.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \\ 2 & -2 \end{pmatrix}.$$

a) Compute the symmetric matrix $S = A^T A$.

$$S = \begin{pmatrix} 9 & -8 \\ -8 & 9 \end{pmatrix}$$

b) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 17 & 0 \\ 0 & 1 \end{pmatrix}$$

c) What is the minimum value of $\|Ax\|$ subject to $\|x\| = 1$? At which vectors is this minimum achieved?

$$\text{minimum value} = 1 \qquad \text{achieved at } x = \pm\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Problem 3.

[20 points]

Consider the difference equation $v_{k+1} = Av_k$ where

$$A = \begin{pmatrix} 0.3 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.3 & 0.6 & 0.3 \end{pmatrix}.$$

a) Compute the characteristic polynomial $p(\lambda)$ of A.

$$p(\lambda) = -\lambda^3 + .8\lambda^2 + .2\lambda$$

b) Find the eigenvalues of A.

[**Hint:** one of the eigenvalues is zero, so you can factor $p(\lambda)$ using the quadratic formula.]

eigenvalues $= 0, -.2, 1$

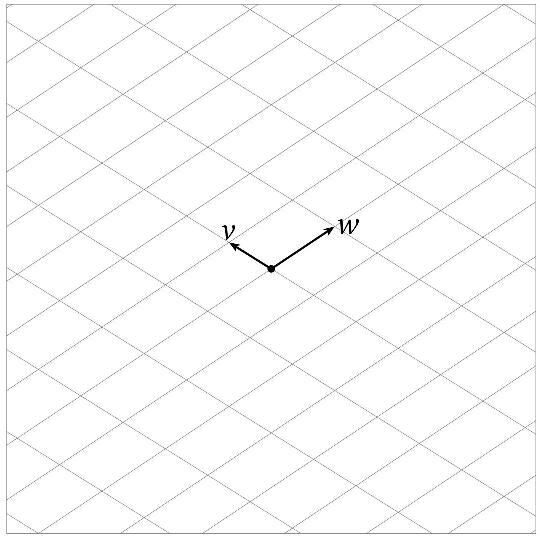
c) Find an eigenbasis $\{w_1, w_2, w_3\}$ for A.

$$w_1 = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} \qquad w_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

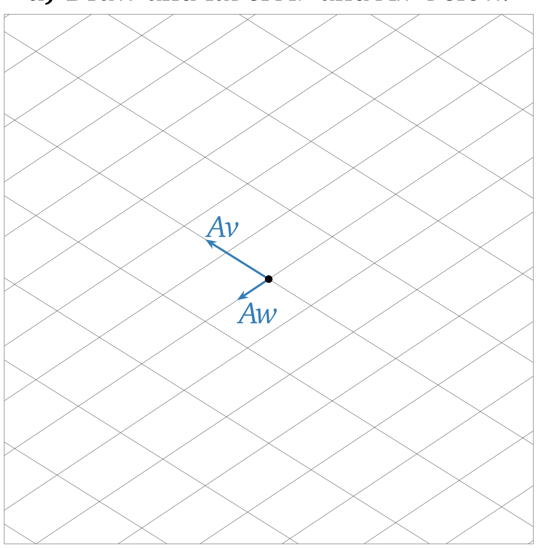
d) If $v_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$ then v_k approaches $x_1 w_1$ as $k \to \infty$.

Problem 4. [10 points]

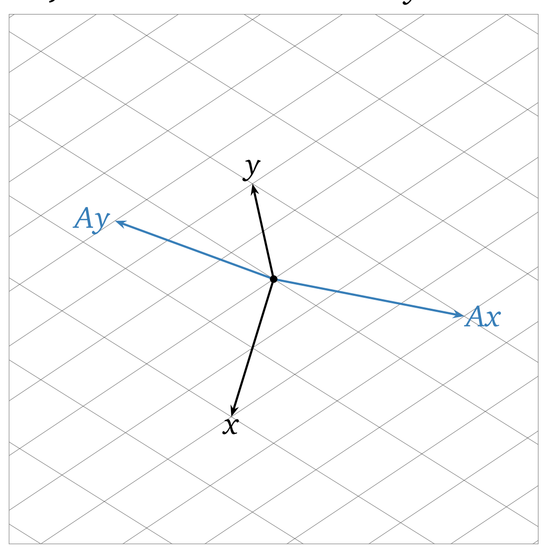
A certain 2×2 matrix A has eigenvectors v and w, pictured below, with corresponding eigenvalues $3/2$ and $-1/2$, respectively.



a) Draw and label Av and Aw below.



b) Draw and label Ax and Ay below.



Problem 5. [20 points]

a) Let A be a matrix that is diagonalizable over the complex numbers. Consider the initial value problem $u' = Au, \; u(0) = u_0 \in \mathbf{R}^n$. What must be true about the eigenvalues of A to guarantee that $u(t) \to 0$ as $t \to \infty$ for every initial value u_0?

The real part of each eigenvalue must be negative.

b) Let V be a plane in $\mathbf{R}^3$. There exists an invertible matrix C such that $P_V = CDC^{-1}$, where D is the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(There is more than one correct answer.)

c) If A is a 2×2 real matrix with complex eigenvalue $a + bi$ $(b \neq 0)$ then

$$\det(A) = a^2 + b^2.$$

d) Suppose that A is diagonalizable. Explain why A^3 is diagonalizable.

If $A = CDC^{-1}$ then $A^3 = CD^3C^{-1}$, and D^3 is still diagonal.

e) Let A be the diagonal matrix $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Find two *different* invertible matrices $C_1 \neq C_2$ and *different* diagonal matrices $D_1 \neq D_2$ such that

$$C_1 D_1 C_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = C_2 D_2 C_2^{-1}.$$

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_1 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Problem 6.

[20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A 2×2 matrix that is neither diagonalizable nor invertible.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

b) A 3×3 matrix with no real eigenvalues.

No such matrix exists: any degree-3 real polynomial has a real root.

c) A matrix having eigenvalue 3 with algebraic multiplicity 1 and geometric multiplicity 2.

No such matrix exists: this violates the AM≥GM theorem.

d) A 2×2 symmetric matrix that does not have a Cholesky decomposition.

Any non-positive-definite diagonal matrix is an example.

e) A 2×2 symmetric matrix that does not have an orthogonal diagonalization QDQ^T.

No such matrix exists by the spectral theorem.