Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

\[
\begin{bmatrix}
\cos 90^\circ & \sin 90^\circ \\
-\sin 90^\circ & \cos 90^\circ \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\end{bmatrix} = \begin{array}{c}
\Phi \\
\Phi \\
\end{array}
\]

[Hint: this is a joke.]
Problem 1. [20 points]

Consider the subspace

a) Find an orthogonal basis for

\[ \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}. \]

Now we change subspaces to avoid carry-through error. Consider the subspace

\[ V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\}, \]

and note that the spanning vectors are orthogonal.

b) Compute the orthogonal projection of \( b = (-7, 4, -4) \) onto \( V \).

\[ b_V = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} \]

c) Compute the matrix \( P_V \) for projection onto \( V \).

\[ P_V = \frac{1}{9} \begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix} \]

d) Compute the matrix \( P_{V^\perp} \) for projection onto \( V^\perp \).

\[ P_{V^\perp} = \frac{1}{9} \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \]

e) Find a basis of \( \text{Nul}(P_V) \).

\[ \left\{ \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} \right\} \]
Problem 2. [15 points]

In this problem we will consider the best-fit line \( y = Cx + D \) through the data points 

\[
\begin{pmatrix}
1 & 1 \\
2 & 1 \\
3 & 1 \\
4 & 1
\end{pmatrix}, \quad \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{pmatrix}.
\]

**a)** The line \( y = Cx + D \) passes through all four points if and only if the matrix equation

\[
\begin{pmatrix}
1 & 1 \\
2 & 1 \\
3 & 1 \\
4 & 1
\end{pmatrix} \begin{pmatrix}
C \\
D
\end{pmatrix} = \begin{pmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{pmatrix}
\]

is satisfied (fill in the blank).

Let \( A \) be the coefficient matrix in the previous problem. In the QR decomposition of \( A \), the matrix \( Q \) is

\[
Q = \begin{pmatrix}
1\sqrt{30}/3 & 2/\sqrt{6} \\
2\sqrt{30}/3 & 1/\sqrt{6} \\
3\sqrt{30}/3 & 0 \\
4\sqrt{30}/3 & -1/\sqrt{6}
\end{pmatrix}.
\]

**b)** Explain why \( R = Q^T A \), and compute \( R \).

Since \( Q^T Q = I_2 \), multiplying both sides of \( A = QR \) by \( Q^T \) gives \( Q^T A = R \).

\[
R = \begin{pmatrix}
\sqrt{30}/30 & \sqrt{6}/30 \\
0 & \sqrt{6}/3
\end{pmatrix}
\]

**c)** Use the QR decomposition to find the best-fit line through the data points

\[
\begin{pmatrix}
1 \\
1 \\
2 \\
-1 \\
4 \\
0
\end{pmatrix}.
\]

\[
y = \frac{1}{2}x + \frac{3}{2}
\]

**d)** Graph the line you found in **c)** below. Explain which quantity was minimized in terms of the graph.

The sum of the squares of the lengths of the red lines is minimized.
Problem 3. \[15 \text{ points}\]

Consider the matrix
\[
A = \begin{pmatrix}
1 & 1 & 0 & -2 \\
-1 & -1 & 1 & 1 \\
3 & 3 & -1 & -5
\end{pmatrix}.
\]

a) Compute bases of all four fundamental subspaces of $A$.

\[
\text{Nul}(A): \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{Col}(A): \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}
\]

\[
\text{Row}(A): \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\} \quad \text{Nul}(A^T): \left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\}
\]

b) Compute the orthogonal decomposition of \((0, 3, 3)\) with respect to $V = \text{Col}(A)$.
\[
\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}
\]

c) Compute the distance from \((0, 3, 3)\) to $\text{Col}(A)$.
\[
\sqrt{6}
\]
Problem 4. [10 points]

For a certain $2 \times 2$ matrix $A$, the row space of $A$ is drawn in the picture on the left, and the column space is drawn in the picture on the right.

\begin{itemize}
  \item[a)] rank($A$) = 1.
  \item[b)] Draw Nul($A$) in the picture on the left.
  \item[c)] Draw Nul($A^T$) in the picture on the right.
  \item[d)] If $V = \text{Col}(A)$ and $b$ is the vector in the picture on the right, draw and label the vectors $b_V$ and $b_{V\perp}$.
\end{itemize}
Problem 5. [20 points]

a) Find a matrix whose null space is Span\{(1, 1, 1)\}.

The orthogonal complement of Span\{(1, 1, 1)\} is Nul\((1 \ 1 \ 1)\), which has basis \{(-1, 1, 0), (-1, 0, 1)\}. Taking orthogonal complements again, we have

\[
\text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \text{Nul}\left(\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\right).
\]

b) For which value(s) of \(k\), if any, do the following vectors not form a basis of \(\mathbb{R}^4\)?

\[
\begin{bmatrix}
1 \\
0 \\
-6
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
1 \\
-1
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
2 \\
k
\end{bmatrix}
\]

Expanding cofactors along the first row, we compute

\[
\det\left(\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
-6 & -1 & -8 & k
\end{pmatrix}\right) = \det\left(\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 3 \\
-1 & -8 & k
\end{pmatrix}\right) + 6\det\left(\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 3 \\
1 & 0 & 2
\end{pmatrix}\right)
\]

\[
= k + 32.
\]

Hence the columns are not linearly independent exactly when \(k = -32\).

c) Which of the following properties of a matrix are not changed by row operations? (Fill in the bubbles of all that apply)

- The rank
- The column space
- The null space
- The row space
- The left null space
- The determinant
- The reduced row echelon form

d) If \(\det(A) = 2\) and \(\det(B) = 3\), compute the following determinants:

\[
\det(A^2) = 4 \quad \det(AB^T) = 6 \quad \det(BA^kB^{-1}) = 2^k
\]

(Here \(A\) and \(B\) are square matrices of the same size and \(k\) is a whole number.)
Problem 6. [20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

a) A $3 \times 2$ matrix $A$ such that $Ax = (1, 2, 3)$ has more than one least-squares solution.
   
   Any $3 \times 2$ matrix that does not have full column rank is an example.

b) A matrix $A$ in RREF satisfying $\dim \text{Row}(A) = 2$ and $\dim \text{Nul}(A) = 3$.
   
   Any RREF matrix with five columns and two nonzero rows is an example.

c) A matrix $Q$ with orthonormal columns, such that $\det(QQ^T) = 0$.
   
   Any non-square matrix with orthonormal columns is an example.

d) A matrix $A$ whose column space $V = \text{Col}(A)$ is a plane in $\mathbb{R}^3$, such that $\text{rank}(P_V) = 1$.
   
   Impossible: $\text{rank}(P_V) = \dim(V) = 2$. 