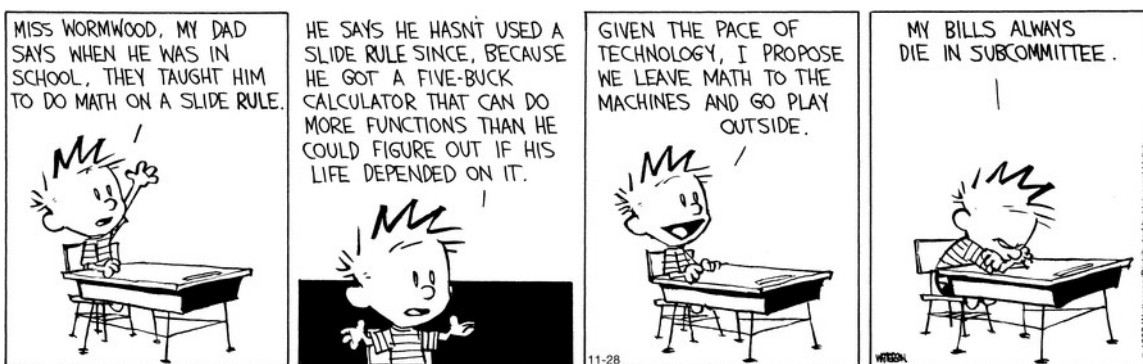


**MATH 218D-1  
MIDTERM EXAMINATION 1**

<b>Name</b>		<b>Duke NetID</b>	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



## Problem 1.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -3 & -2 \\ 9 & 7 & 2 \end{pmatrix}.$$

- a) Find a lower-unitriangular matrix  $L$  and a matrix  $U$  in REF such that  $A = LU$ .

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

- b) Use the LU decomposition you found in a) to solve the equation  $Ax = (0, 2, 0)$ .

$$x = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

- c) Briefly explain why one would want to use an LU decomposition to solve  $Ax = b$ .

Solving  $LUx = b$  using forward-substitution and back-substitution is much faster than running elimination on  $(A | b)$ .

- d) Express  $L$  as a product of three elementary matrices.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

## Problem 2.

[25 points]

Consider the system of equations

$$\begin{aligned}x_1 - 2x_2 - x_4 &= 2 \\ -x_1 + 2x_2 + x_3 + 2x_4 &= -3 \\ 3x_1 - 6x_2 - x_3 - 4x_4 &= 7.\end{aligned}$$

a) Express this system of equations as a matrix equation  $Ax = b$ :

$$\begin{pmatrix} 1 & -2 & 0 & -1 \\ -1 & 2 & 1 & 2 \\ 3 & -6 & -1 & -4 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix}$$

b) Perform Gauss–Jordan elimination on the augmented matrix  $(A | b)$  to produce a matrix in reduced row echelon form. Please write all row operations that you perform.

$$\left( \begin{array}{cccc|c} 1 & -2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

c) The free variables(s) are  $x_2$  and  $x_4$  and the rank of  $A$  is 2

d) Write the solution set of this system of equations in parametric vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

e) The solution set is a (circle one)  $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$  in (fill in the blank)  $\mathbf{R}^4$ .

f) Find a basis for  $\text{Nul}(A)$ .

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

g) Write down any *nontrivial* solution of  $Ax = 0$ .

Any nonzero linear combination of the vectors in **f)** works. For instance,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

### Problem 3.

[20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}.$$

a) Find a linear relation among  $v_1, v_2, v_3, v_4$ .

Any nonzero vector in Problem 2 f) gives the weights of a linear relation. For instance,

$$0 = v_1 + 0v_2 - v_3 + v_4$$

b) The set  $\{v_1, v_2, v_3, v_4\}$  is (circle one)  $\begin{pmatrix} \text{linearly dependent} \\ \text{linearly independent} \end{pmatrix}$ .

c) Which one of the following two vectors is in  $\text{Span}\{v_1, v_2, v_3, v_4\}$ ?

(Circle the correct one.)

$$u = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

d) Express  $(2, -3, 7)$  as a linear combination of  $v_1, v_2, v_3, v_4$  in two different ways.

We can take  $(x_2, x_4) = (1, 0)$  and  $(x_2, x_4) = (0, 1)$  in Problem 2 d):

$$\begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$
$$\begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 2 \\ -6 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

e) Which of the following sets form a basis for  $\text{Span}\{v_1, v_2, v_3, v_4\}$ ?

(Circle all that apply.)

$$\{v_1, v_2\} \quad \{v_1, v_2, v_3\} \quad \{v_2, v_3\}$$

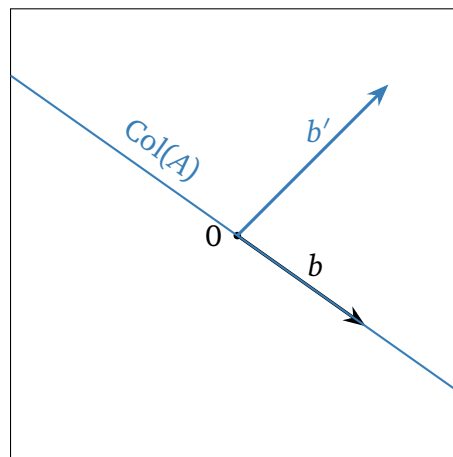
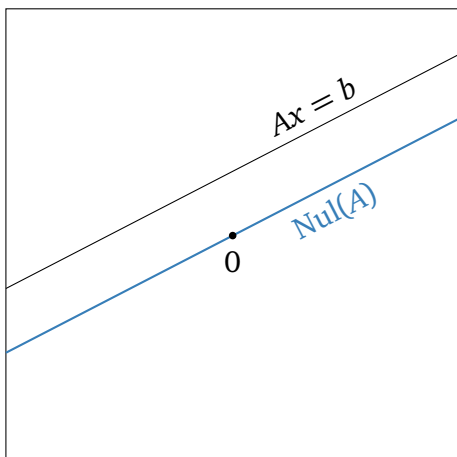
$$\{v_2, v_3, v_4\} \quad \{v_3, v_4\} \quad \{v_2, v_4\}$$

f)  $\dim(\text{Span}\{v_1, v_2, v_3, v_4\}) = 2$

### Problem 4.

[15 points]

For a certain  $2 \times 2$  matrix  $A$  and vector  $b \in \mathbf{R}^2$ , the solution set of  $Ax = b$  is drawn in the picture on the left, and  $b$  is drawn in the picture on the right.



- a) Draw  $\text{Nul}(A)$  in the picture on the left.
- b) Draw  $\text{Col}(A)$  in the picture on the right.
- c) Draw a vector  $b'$  in the picture on the right that makes the system  $Ax = b'$  inconsistent.

## Problem 5.

[10 points]

For each of the following subsets,

- determine if the subset is a subspace.

If it is not a subspace,

- explain why not.

If it is a subspace,

- express it as the null space or the column space of a matrix, and
- find a basis.

a) 
$$V_1 = \left\{ \begin{pmatrix} 2(a+c)-b \\ a-2b+c \end{pmatrix} : a, b, c \in \mathbf{R} \right\}$$

This is a subspace:

$$V_1 = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} = \text{Col} \begin{pmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \end{pmatrix}.$$

In fact,  $V_1 = \mathbf{R}^2$ , so any two noncollinear vectors in the plane form a basis. For instance,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

b) 
$$V_2 = \left\{ \begin{pmatrix} a+b \\ 2 \\ 2a-2b \end{pmatrix} : a, b, c \in \mathbf{R} \right\}$$

This is not a subspace: it does not contain 0.

c) 
$$V_3 = \text{the } x\text{-axis in the } xy\text{-plane}$$

This is a subspace:

$$V_3 = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \text{Col} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Moreover,  $\{(1, 0)\}$  is a basis.

d) 
$$V_4 = \text{all vectors } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } x \geq y$$

This is not a subspace. It is not closed under scalar multiplication:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in V_4 \quad \text{but} \quad -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin V_4.$$

## Problem 6.

[15 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead. You need not justify your answers.

- a) A  $2 \times 2$  matrix whose column space is a point.

The only answer is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- b) A  $2 \times 2$  matrix whose null space is *all of*  $\mathbf{R}^2$ .

The only answer is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- c) An invertible  $3 \times 3$  matrix  $A$  such that the solution set of  $Ax = 0$  is the  $x$ -axis.

As if.

- d) A  $2 \times 2$  matrix whose columns are linearly *dependent*.

One example is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- e) A  $2 \times 2$  matrix  $A$  such that the solution set of  $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  is the point  $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ , and such that the system  $Ax = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is inconsistent.

Yeah, right.