Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a four-function calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
0 & 2 & -2 & -2 \\
2 & 0 & 8 & 0 \\
-3 & -1 & -3 & 6 \\
6 & 0 & 12 & -6
\end{pmatrix}.
\]

a) Perform Gaussian elimination with maximal partial pivoting to obtain a \( PA = LU \) decomposition of \( A \). You should end up with

\[
U = \begin{pmatrix}
6 & 0 & 12 & -6 \\
0 & 2 & -2 & -2 \\
0 & 0 & 4 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Please write the row operations you performed. (You can continue your work on the back of this sheet.)

\[
L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1/3 & 0 & 1 & 0 \\
-1/2 & -1/2 & 1/2 & 1
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

b) Briefly explain the reason one might want to always choose the largest pivot in absolute value.

Maximal partial pivoting reduces rounding errors when implemented on a computer.
Problem 2. [20 points]

Consider the matrix
\[
A = \begin{pmatrix}
1 & 2 & -1 \\
-1 & -3 & 4 \\
-2 & -6 & 9
\end{pmatrix}.
\]

a) Compute \(A^{-1}\). Please write the row operations you performed.
\[
A^{-1} = \begin{pmatrix}
3 & 12 & -5 \\
-1 & -7 & 3 \\
0 & -2 & 1
\end{pmatrix}
\]

b) Express \(A^{-1}\) as a product of elementary matrices. (Your answer will be a product of matrices with numbers in them, as opposed to row operations.)
\[
A^{-1} = \begin{pmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

c) Solve \(Ax = b\), where \(b = (b_1, b_2, b_3)\) is an unknown vector. (Your answer will be a formula in \(b_1, b_2, b_3\).)
\[
x = \begin{pmatrix}
3b_1 + 12b_2 - 5b_3 \\
-b_1 - 7b_2 + 3b_3 \\
-2b_2 + b_3
\end{pmatrix}
\]
Problem 3. [20 points]

Consider the system of equations
\[
\begin{align*}
x_1 + 2x_2 - x_3 - x_4 &= 2 \\
x_2 + x_3 + 2x_4 &= 1.
\end{align*}
\]

a) Express the solution set as a translate of a span:
\[
\text{solution set} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \right\}
\]

b) The solution set is a (circle one) \( \text{point} \), \( \text{line} \), \( \text{plane} \) in (fill in the blank) \( \mathbb{R}^4 \).

c) The solution set of \( Ax = 0 \) has dimension 2.

d) Describe \( \text{Span}\{(-1)^1, (2)^1, (-1)^1, (2)^1\} \) geometrically:
\[
\text{it is a (circle one) } \begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \end{pmatrix} \text{ in (fill in the blank) } \mathbb{R}^2.
\]

e) Find numbers \( b_1, b_2 \) making the system inconsistent. If no such numbers exist, explain why.

No such numbers exist. The columns of the coefficient matrix of this system span all of \( \mathbb{R}^2 \).
Problem 4. [12 points]

Give examples of $2 \times 2$ matrices $A, B, C$ with ranks 0, 1, and 2, respectively. Draw pictures of the solution set of $Ax = 0$ and the span of the columns of $A$, and likewise for $B$ and $C$. (Recall that the rank of a matrix is the number of pivots.) Be precise!

a) Rank 0: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

b) Rank 1: $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

c) Rank 2: $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(There are many possible answers for $B$ and $C$, although the pictures will be the same for any choice of $C$.)

Problem 5. [10 points]

A certain $2 \times 2$ matrix $A$ has columns $v$ and $w$, pictured below. Solve the equations $Ax_1 = b_1$ and $Ax_2 = b_2$, where $b_1$ and $b_2$ are the vectors in the picture.

$x_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$x_2 = \begin{pmatrix} -3/2 \\ -2 \end{pmatrix}$
Problem 6.

Short-answer questions: you do not need to justify your answers.

a) Suppose that $A$ is a $4 \times 2$ matrix such that the solution set of $A(x, y) = 0$ is the line $y = x$. Let $b$ be a nonzero vector in $\mathbb{R}^4$. Which of the following are definitely not the solution set of $Ax = b$? (Circle all that apply.)

- The line $y = x$.
- The $y$-axis.
- The line $y = x + 1$.
- The point $(1, 2)$.
- The empty set.

b) Consider the following plane in $\mathbb{R}^3$:

$$P = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$  

Find two other vectors that span $P$. Your answer cannot contain a scalar multiple of $(1, 0, -1)$ or $(1, -1, 0)$.

$$P = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$  

(Any noncollinear vectors whose coordinates sum to zero will work.)

c) Find three vectors $u, v, w \in \mathbb{R}^3$ such that $\text{Span}\{u, v, w\}$ is a plane, but such that $w \notin \text{Span}\{u, v\}$.

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$  

(The vectors $u$ and $v$ must be collinear.)

d) A nonzero $2 \times 3$ matrix $A$ has the property that $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is inconsistent. The span of the columns of $A$ is a

- point
- line
- plane
- space.