# MATH 218D-1
# PRACTICE MIDTERM EXAMINATION 1

| Name | | Duke NetID | |
|---|---|---|---|

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

## Problem 1. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & 8 & 0 \\ -3 & -1 & -3 & 6 \\ 6 & 0 & 12 & -6 \end{pmatrix}.$$

**a)** Perform Gaussian elimination with maximal partial pivoting to obtain a $PA = LU$ decomposition of $A$. You should end up with

$$U = \begin{pmatrix} 6 & 0 & 12 & -6 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Please write the row operations you performed. (You can continue your work on the back of this sheet.)

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1 & 0 \\ -1/2 & -1/2 & 1/2 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**b)** Briefly explain the reason one might want to always choose the largest pivot in absolute value.

Maximal partial pivoting reduces rounding errors when implemented on a computer.

## Problem 2. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & 4 \\ -2 & -6 & 9 \end{pmatrix}.$$

**a)** Compute $A^{-1}$. Please write the row operations you performed.

$$A^{-1} = \begin{pmatrix} 3 & 12 & -5 \\ -1 & -7 & 3 \\ 0 & -2 & 1 \end{pmatrix}$$

**b)** Express $A^{-1}$ as a product of elementary matrices. (Your answer will be a product of matrices with numbers in them, as opposed to row operations.)

$$A^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**c)** Solve $Ax = b$, where $b = (b_1, b_2, b_3)$ is an unknown vector. (Your answer will be a formula in $b_1, b_2, b_3$.)

$$x = \begin{pmatrix} 3b_1 + 12b_2 - 5b_3 \\ -b_1 - 7b_2 + 3b_3 \\ -2b_2 + b_3 \end{pmatrix}$$

## Problem 3.

[20 points]

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 - \phantom{2}x_4 &= 2 \\ x_2 + x_3 + 2x_4 &= 1. \end{aligned}$$

**a)** Express the solution set as a translate of a span:

$$\text{solution set} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

**b)** The solution set is a (circle one) $\begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^4$.

**c)** The solution set of $Ax = 0$ has dimension 2.

**d)** Describe $\text{Span}\left\{ \binom{1}{0}, \binom{2}{1}, \binom{-1}{1}, \binom{-1}{2} \right\}$ geometrically:

it is a (circle one) $\begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^2$.

**e)** Find numbers $b_1, b_2$ making the system

$$\begin{aligned} x_1 + 2x_2 - x_3 - \phantom{2}x_4 &= b_1 \\ x_2 + x_3 + 2x_4 &= b_2 \end{aligned}$$

inconsistent. If no such numbers exist, explain why.

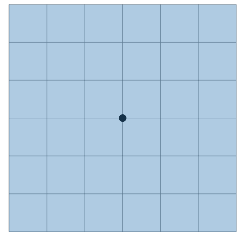
No such numbers exist. The columns of the coefficient matrix of this system span all of $\mathbf{R}^2$.

## Problem 4. [12 points]

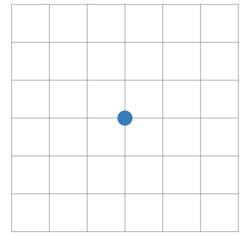
Give examples of $2 \times 2$ matrices $A, B, C$ with ranks 0, 1, and 2, respectively. Draw pictures of the solution set of $Ax = 0$ and the span of the columns of $A$, and likewise for $B$ and $C$. (Recall that the *rank* of a matrix is the number of pivots.) Be precise!

**a)** Rank 0: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$Ax = 0$

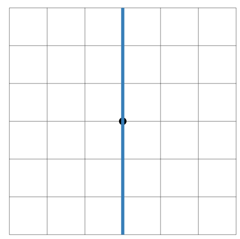


Span{cols of $A$}

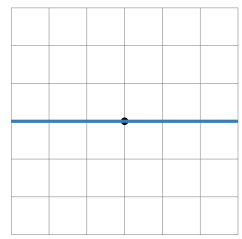


**b)** Rank 1: $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$Bx = 0$

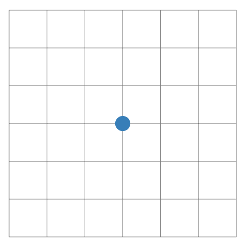


Span{cols of $B$}

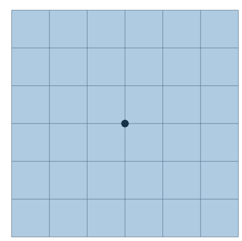


**c)** Rank 2: $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$Cx = 0$



Span{cols of $C$}



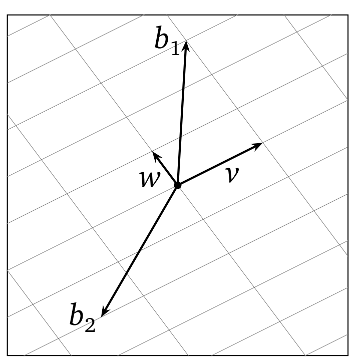
(There are many possible answers for $B$ and $C$, although the pictures will be the same for any choice of $C$.)

## Problem 5. [10 points]

A certain $2 \times 2$ matrix $A$ has columns $v$ and $w$, pictured below. Solve the equations $Ax_1 = b_1$ and $Ax_2 = b_2$, where $b_1$ and $b_2$ are the vectors in the picture.

$$x_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} -3/2 \\ -2 \end{pmatrix}$$



## Problem 6. [16 points]

Short-answer questions: you do not need to justify your answers.

**a)** Suppose that $A$ is a $4 \times 2$ matrix such that the solution set of $A\binom{x}{y} = 0$ is the line $y = x$. Let $b$ be a *nonzero* vector in $\mathbf{R}^4$. Which of the following are definitely *not* the solution set of $Ax = b$? (Circle all that apply.)

The line $y = x$. The $y$-axis. The line $y = x + 1$.

The point $(1, 2)$. The empty set.

**b)** Consider the following plane in $\mathbf{R}^3$:

$$P = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Find two *other* vectors that span $P$. Your answer cannot contain a scalar multiple of $(1, 0, -1)$ or $(1, -1, 0)$.

$$P = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

(Any noncollinear vectors whose coordinates sum to zero will work.)

**c)** Find three vectors $u, v, w \in \mathbf{R}^3$ such that $\mathrm{Span}\{u, v, w\}$ is a *plane*, but such that $w \notin \mathrm{Span}\{u, v\}$.

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(The vectors $u$ and $v$ must be collinear.)

**d)** A *nonzero* $2 \times 3$ matrix $A$ has the property that $Ax = \binom{1}{1}$ is inconsistent. The span of the columns of $A$ is a

point line plane space.