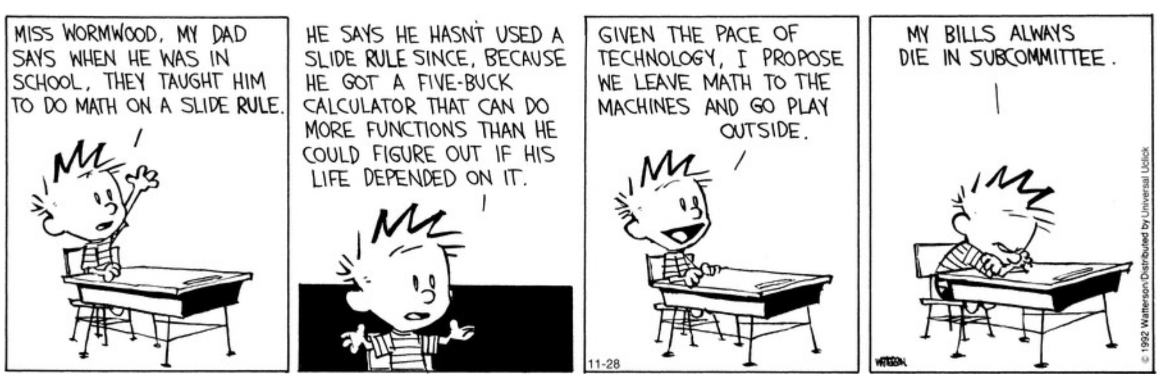
MATH 218D-1
MAKE-UP MIDTERM EXAMINATION 1

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 \\ -4 & -5 & 1 \\ 12 & 7 & 3 \end{pmatrix}.$$

a) Find a lower-unitriangular matrix L and a matrix U in REF such that $A = LU$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 4 & 3 & 2 \\ 0 & -2 & 3 \\ 0 & 0 & -6 \end{pmatrix}$$

b) Use the LU decomposition you found in **a)** to solve the equation $Ax = (3, 2, 8)$.

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

c) Briefly explain why one would want to use an LU decomposition to solve $Ax = b$.

Solving $LUx = b$ using forward-substitution and back-substitution is much faster than running elimination on $\left(A \mid b\right)$.

d) Express L as a product of three elementary matrices.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Problem 2. [25 points]

Consider the system of equations

$$\begin{aligned} x_1 \phantom{{}+ x_2} + x_3 + x_4 &= 3 \\ -x_1 + x_2 + 3x_3 - x_4 &= -2 \\ x_1 + 2x_2 + 9x_3 + 2x_4 &= 5. \end{aligned}$$

a) Express this system of equations as a matrix equation $Ax = b$:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 3 & -1 \\ 1 & 2 & 9 & 2 \end{pmatrix} x = \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix}$$

b) Perform Gauss–Jordan elimination on the augmented matrix $\left(A \mid b\right)$ to produce a matrix in reduced row echelon form. Please write all row operations that you perform.

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

c) The free variables(s) are x_3 and the rank of A is 3.

d) Write the solution set of this system of equations in parametric vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

e) The solution set is a (circle one) $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^4$.

f) Find a basis for Nul(A).

$$\left\{ \begin{pmatrix} -1 \\ -4 \\ 1 \\ 0 \end{pmatrix} \right\}$$

g) Write down any *nontrival* solution of $Ax = 0$.

Any nonzero multiple of the vector in **f)** works. For instance,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

Problem 3. [20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

a) Find a linear relation among v_1, v_2, v_3, v_4.

Any nonzero vector in Problem 2 **f)** gives the weights of a linear relation. For instance,

$$0 = -v_1 - 4v_2 + v_3 + 0v_4$$

b) The set $\{v_1, v_2, v_3, v_4\}$ is (circle one) $\begin{pmatrix} \text{linearly dependent} \\ \text{linearly independent} \end{pmatrix}$.

c) Which of the following vectors is in $\text{Span}\{v_1, v_2, v_3, v_4\}$?

(Circle all that apply.)

$$u = \begin{pmatrix} 2 \\ -6 \\ -5 \end{pmatrix} \qquad w = \begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix}$$

d) Express $(3, -2, 5)$ as a linear combination of v_1, v_2, v_3, v_4 in two different ways.

We can take $x_3 = 0$ and $x_3 = 1$ in Problem 2 **d)**:

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

e) Which of the following sets form a basis for $\text{Span}\{v_1, v_2, v_3, v_4\}$?

(Circle all that apply.)

$$\{v_1, v_2\} \qquad \{v_1, v_2, v_3\} \qquad \{v_1, v_2, v_4\}$$

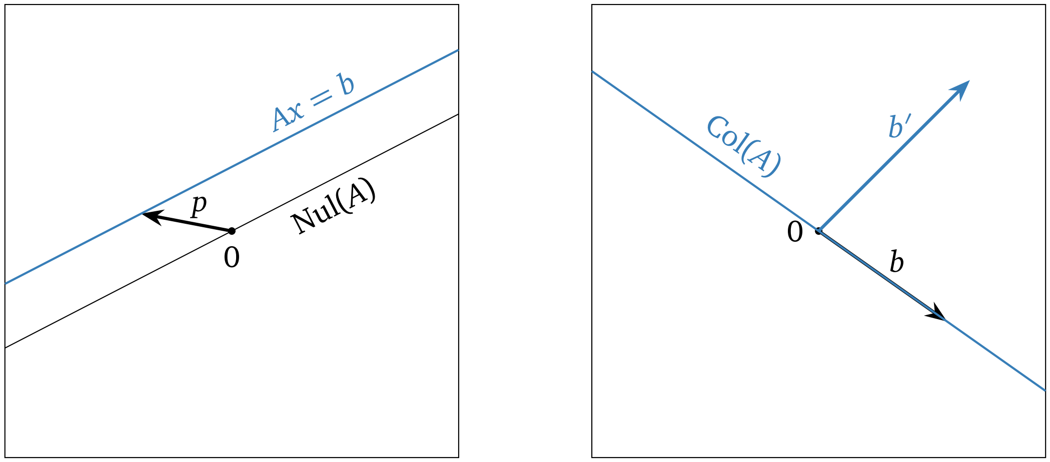
$$\{v_3, v_4\} \qquad \{v_2, v_3, v_4\} \qquad \{v_2, v_4\}$$

f) $\dim\big(\text{Span}\{v_1, v_2, v_3, v_4\}\big) = 3$

Problem 4.

[15 points]

For a certain 2×2 matrix A and vector $p \in \mathbf{R}^2$, the vector p and the null space of A is drawn in the picture on the left, and the vector $b = Ap$ is drawn in the picture on the right.



a) Draw the solution set of $Ax = b$ in the picture on the left.

b) Draw Col(A) in the picture on the right.

c) Draw a vector b' in the picture on the right that makes the system $Ax = b'$ inconsistent.

Problem 5.

[10 points]

For each of the following subsets,

- determine if the subset is a subspace.

If it is not a subspace,

- explain why not.

If it is a subspace,

- express it as the null space or the column space of a matrix, and
- find a basis.

a)

$$V_1 = \left\{ \begin{pmatrix} a \\ a+1 \end{pmatrix} \ : \ a \in \mathbf{R} \right\}$$

This is not a subspace: it does not contain the zero vector.

b)

$$V_2 = \left\{ \begin{pmatrix} a+b \\ 0 \\ 2a-2b \end{pmatrix} \ : \ a, b, c \in \mathbf{R} \right\}$$

This is a subspace:

$$V_1 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\} = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 2 & -2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

In fact, $V_1 = \{(x, 0, z) \ : \ x, z \in \mathbf{R}\}$, so any two noncollinear vectors in this plane form a basis. For instance,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

c)

$$V_3 = \text{the } y\text{-axis in the } xy\text{-plane}$$

This is a subspace:

$$V_3 = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \mathrm{Col}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Moreover, $\{(0, 1)\}$ is a basis.

d)

$$V_4 = \text{all vectors } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } x + 2 \le y$$

This is not a subspace. It is not closed under scalar multiplication:

$$\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \in V_4 \quad \text{but} \quad -\begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \notin V_4.$$

Problem 6. [15 points]

In each part, find an example of a matrix with the given property. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A 2×2 matrix whose column space is *all of* $\mathbf{R}^2$.

One answer is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A 2×2 matrix whose null space is a point.

One answer is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

c) An invertible 3×3 matrix A such that the solution set of $Ax = 0$ is the xy-plane.

As if.

d) A 2×2 matrix whose rows are linearly *dependent*.

One example is

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

e) A 2×2 matrix A such that the solution set of $Ax = \binom{1}{2}$ is the point $\left\{\binom{-1}{1}\right\}$, and such that the solution set of $Ax = \binom{2}{1}$ is a line.

Yeah, right.