1. **Orthogonal matrices**

A *orthogonal matrix* is a *square* matrix $Q$ whose columns form an orthonormal set. Alternately, it is a square matrix $Q$ such that $Q^T Q = I_n$.

a) Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ an orthogonal matrix?

b) Is $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ an orthogonal matrix?

c) Is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ an orthogonal matrix?
2. Rotation and reflection

A rotation matrix $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is an example of an orthogonal matrix.

a) Confirm that $R_\theta$ is an orthogonal matrix by checking $R_\theta^T R_\theta = I_2$.

b) Draw the vectors $R_{\pi/6} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $R_{\pi/6} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

c) Using dot products, compute the angle between the rotated vectors $R_{\pi/6} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $R_{\pi/6} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Confirm that this is the same as the angle between the two vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is an example of a general phenomenon: multiplying by an orthogonal matrix preserves angles and lengths.

Consider a line $L = \text{Span}\{v\} \subset \mathbb{R}^3$, and the orthogonal complement plane $V = L^\perp$. The reflection matrix for reflection across $V$ is the orthogonal matrix $Q = I_3 - 2P_L$, where $P_L$ is the projection matrix for $L$.

d) When $L = \text{Span}\{(0,1,0)\}$, compute the reflection matrix $Q$. Draw the line $L$ and the plane $V$. Compute and draw the vector $(1,1,0)$, the projection $P_L(1,1,0)$, and the reflection $Q(1,1,0)$.

e) Confirm that any reflection matrix $Q = I_3 - 2P_L$ is an orthogonal matrix by showing that $Q^T Q = (I_3 - 2P_L)^T (I_3 - 2P_L) = I_3$.

**Hint:** Remember that $P_L^2 = P_L$ and $P_L^T = P_L$. 
3. **Gram-Schmidt and QR**

The purpose of the Gram–Schmidt process is to replace a basis \( \{v_1, \cdots, v_k\} \) of a subspace \( V \subset \mathbb{R}^n \) with an **orthogonal basis** of \( V \) (a basis whose vectors are an orthogonal set).

The vectors \( v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) are a basis for a plane \( V \subset \mathbb{R}^3 \). Set

\[
u_1 = v_1, \quad u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.\]

These two vectors are the output of the Gram–Schmidt process.

**a)** Compute \( \frac{u_1}{\|u_1\|} \) and \( \frac{u_2}{\|u_2\|} \), and confirm that \( \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|} \right\} \) is an orthonormal set of vectors (you need to compute 3 dot products).

**b)** We can find the QR decomposition of \( A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \) by setting

\[
Q = \begin{pmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \\ \end{pmatrix},
\]

a \( 3 \times 2 \) matrix. Now, \( A = QR \) for some upper-triangular matrix \( R \), and you saw a formula for \( R \) in lecture. Here is another way to find \( R \):

\[
R = Q^T A.
\]

Use this to compute \( R \), and confirm that \( A = QR \) by multiplying \( Q \) times \( R \).

**Note:** The method of finding \( R \) given in lecture is much faster, as it involves only book-keeping your work from finding \( Q \).

**c)** Explain why this formula for \( R \) worked, i.e. why \( A = QR \) had to imply that \( Q^T A = R \).

**Hint:** Multiply both sides of \( A = QR \) by \( Q^T \). What does \( Q^T Q \) always equal, for a matrix \( Q \) with orthonormal columns?
4. Least squares

We want to find the line \( y = Cx + D \) which best fits the data points \((1, 3), (2, 2), (-2, 1)\) (in the least-squares sense). If there were a line which was an exact fit, the coefficients \( C \) and \( D \) would solve the equation

\[
A \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ where } A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}.
\]

But there is no solution to this, as these 3 data points are not collinear. Instead, we'll find the least-squares solution \( \hat{x} = \begin{pmatrix} C \\ D \end{pmatrix} \), i.e. the solution to

\[
A^T A \hat{x} = A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.
\]

a) Compute \( A^T A \) and \( A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \), and solve for the least-squares solutions \( \hat{x} = \begin{pmatrix} C \\ D \end{pmatrix} \).

b) Plot the data points and the least-squares line \( y = Cx + D \).

c) What do the numbers in the vector \( A\hat{x} \) mean?

d) Compute the error \( \left\| A\hat{x} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\|^2 \).

e) You already found the \( QR \) decomposition for this matrix \( A \) in problem 3. Solve the equation

\[
R \hat{x} = Q^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},
\]

and confirm that this \( \hat{x} \) is the same vector you found in part a).
5. Another Gram–Schmidt

a) Apply the Gram–Schmidt process to the vectors \( v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1) \) to obtain an orthogonal set \( u_1, u_2, u_3 \).

(Recall that \( u_1 = v_1, u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1, u_3 = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 \).)

b) Find the QR decomposition of \( A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \).

c) Consider the vector \( b = (1, 1, 1) \). Since \( \{u_1, u_2, u_3\} \) is a basis for \( \mathbb{R}^3 \), there are scalars \( x_1, x_2, x_3 \) such that \( b = x_1 u_1 + x_2 u_2 + x_3 u_3 \). Solve for these scalars by taking the dot product of this equation with each of \( u_1, u_2, u_3 \), giving 3 equations

\[ b \cdot u_i = (x_1 u_1 + x_2 u_2 + x_3 u_3) \cdot u_i \text{ for } i = 1, 2, 3. \]

(These equations simplify dramatically when you compute the dot products.)

d) Explain how you could instead solve for these scalars using the formula \( QQ^T = P_{\mathbb{R}^3} = I_3 \).

**Hint**: First, \( Q(Q^T b) = b \). Second, \( Q \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a_1 \frac{u_1}{\|u_1\|} + a_2 \frac{u_2}{\|u_2\|} + a_3 \frac{u_3}{\|u_3\|}) \).