1. **Projection onto a line**
   For each of the following,
   (1) project the vector $b$ onto the line $V = \text{Span}\{v\}$;
   (2) draw the three vectors $b, b_V, b_{V\perp}$.
   
   a) $b = (1, 1), \ v = (1, 0)$
   
   b) $b = (0, 2), \ v = (1, 1)$
   
   c) $b = (1, 2, 3), \ v = (1, 1, -1)$.

2. **Planes and normal vectors**
   The subspace $V = \text{Span}\{(1, 1, 2), (1, 3, 1)\}$ of $\mathbb{R}^3$ is a plane.
   
   a) Make the vectors $(1, 1, 2), (1, 3, 1)$ into the rows of a $2 \times 3$ matrix $A$ - this means that $\text{Row}(A) = V$. Find a basis for $\text{Nul}(A)$. Since
   
   \[ V^\perp = \text{Row}(A)^\perp = \text{Nul}(A), \]
   
   you have found a basis $v = (a, b, c)$ for the line $V^\perp$.
   
   In other words, you have found a basis for $V^\perp$ by solving the two orthogonality equations
   \[
   (a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0, \\
   (a, b, c) \cdot (1, 3, 1) = a + 3b + c = 0.
   \]
   
   b) Confirm that $V$ is the plane \{$(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0$\}, by showing that both $(1, 1, 2)$ and $(1, 3, 1)$ solve this equation. *The coefficients of a plane’s equation make a normal vector for the plane.*
   
   c) Find the orthogonal decomposition $b = b_V + b_{V\perp}$ of the vector $b = (1, 1, 1)$ with respect to the plane $V$ and the orthogonal line $V^\perp$.
   
   **Hint:** It is easier to compute $b_{V\perp}$, as it is the projection of $b$ onto the line $V^\perp$ spanned by the vector $v = (a, b, c)$. 
3. Projection onto a plane

Consider the plane

\[ V = \text{Span}\{(1, 1, 1, 1), (1, 2, 3, 4)\} \]

in \( \mathbb{R}^4 \). We will find the orthogonal projection of \( b = (1, -1, -3, -5) \) onto \( V \). This is a vector \( b_V \in \mathbb{R}^4 \) so that \( b_V \in V \) and \( b_{V^\perp} = b - b_V \in V^\perp \).

Since \( b_V \) is in \( V \), it must equal

\[ b_V = \hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4) \]

for some scalars \( \hat{x}_1 \) and \( \hat{x}_2 \). **We will compute the orthogonal projection by solving for these scalars.**

The vector \( b_{V^\perp} \) is orthogonal to every vector in \( V \), in particular it is orthogonal to both \( (1, 1, 1, 1) \) and \( (1, 2, 3, 4) \). We get two equations:

\[
(1, 1, 1, 1) \cdot b_{V^\perp} = 0, \\
(1, 2, 3, 4) \cdot b_{V^\perp} = 0.
\]

Expanding \( b_{V^\perp} = b - b_V = (1, -1, -3, -5) - (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) \), we can rewrite these two equations as

\[
(1, 1, 1, 1) \cdot (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) = (1, 1, 1, 1) \cdot (1, -1, -3, -5), \\
(1, 2, 3, 4) \cdot (\hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4)) = (1, 2, 3, 4) \cdot (1, -1, -3, -5).
\]

**a)** By computing the dot-products, convert this into two linear equations in the two unknowns \( \hat{x}_1 \) and \( \hat{x}_2 \).

**b)** Solve for \( \hat{x}_1 \) and \( \hat{x}_2 \), and compute the orthogonal projection

\[ b_V = \hat{x}_1(1, 1, 1, 1) + \hat{x}_2(1, 2, 3, 4). \]

**c)** Confirm that the vector \( b_{V^\perp} = b - b_V \) is orthogonal to \( V \) by checking that

\[ b_{V^\perp} \cdot (1, 1, 1, 1) = 0 \text{ and } b_{V^\perp} \cdot (1, 2, 3, 4) = 0. \]

**d)** Write down a matrix \( A \) whose column are the two vectors which span \( V \), and compute \( A^T A \), the “matrix of dot products”. Compute the vector \( A^T b \). Explain where the matrix equation \( A^T A \tilde{x} = A^T b \) (the **normal equation**) appears in **a)**-**b)**, and also where the product \( b_V = A\tilde{x} \) appears.

**e)** Compute the projection matrix \( P = A(A^T A)^{-1}A^T \) for the subspace \( V \) – this is the matrix which, when multiplied with \( b \), produces the projection \( b_V \): \( Pb = b_V \).

We’ll discuss projection matrices in next week’s lectures.

**f)** Compute the vectors \( (I_4 - P) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) and \( (I_4 - P) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \). Explain why these two vectors give a basis for the plane \( V^\perp \).
g) Use your answer to f) to describe the plane $V$ via two implicit equations:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0 \text{ and } c'_1 x_1 + c'_2 x_2 + c'_3 x_3 + c'_4 x_4 = 0\}.$$ 

In other words, what coefficient vectors $(c_1, c_2, c_3, c_4)$ and $(c'_1, c'_2, c'_3, c'_4)$ can we use to describe $V$, and why? Confirm that every vector in $V$ satisfies these equations by checking that both $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ do.

4. **Projection matrices for lines**
For each of the following lines $L$, compute the projection matrix $P_L$.

- a) $L = \text{Span}\{(1, 1)\}$, 
- b) $L = \text{Span}\{(1, 2, 3)\}$, 
- c) $L = \{(x, y, z) \in \mathbb{R}^3 : 2x + y + z = 0\}^\perp$.

5. **Projection matrices for planes**
Consider the plane

$$V = \text{Span}\{(1, 1, 1, 1), (1, 2, 3, 4)\}$$

in $\mathbb{R}^4$.

- a) Compute the projection matrix $P_v$ for the subspace $V$ – this is the matrix which, when multiplied with a vector $b$, produces the projection $b_v$:

$$P_v b = b_v.$$ 

(Feel free to use a computer to help with the matrix multiplications in the formula $P_v = A(A^T A)^{-1} A^T$ if you are finding it tedious.)

- b) Compute the vectors $(I_4 - P_v) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $(I_4 - P_v) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Explain why these two vectors give a basis for the plane $V^\perp$.

- c) Use your answer to b) to describe the plane $V$ via two implicit equations:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0 \text{ and } c'_1 x_1 + c'_2 x_2 + c'_3 x_3 + c'_4 x_4 = 0\}.$$ 

In other words, what coefficient vectors $(c_1, c_2, c_3, c_4)$ and $(c'_1, c'_2, c'_3, c'_4)$ can we use to describe $V$, and why? Confirm that every vector in $V$ satisfies these equations by checking that both $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ do.
6. Some mistakes to avoid

A false “fact”: every projection matrix \( P = A(A^T A)^{-1} A^T \) equals the identity matrix \( I \).

A false “proof”:
\[
P = A(A^T A)^{-1} A^T = A A^{-1} (A^T)^{-1} A^T = (A A^{-1})(A^T)^{-1} A^T = I \cdot I = I.
\]

a) What is wrong with this proof?

b) In what case would this proof be correct?

Consider the subspace \( V = \text{Span}\{(1, 1, 1, -1), (2, 1, 1, 2), (3, 2, 2, 1)\} \) in \( \mathbb{R}^4 \). \( V \) is the column space of the matrix
\[
A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.
\]

c) It would be incorrect to say that \( P = A(A^T A)^{-1} A^T \) is the projection matrix for \( V \). Why?

**Hint:** Try computing \( P \) - what goes wrong?

d) Find a matrix \( B \) so that \( P = B(B^T B)^{-1} B^T \) is the projection matrix for \( V \) - you do not need to compute \( B \).