Math 218D Problem Session
Week 2

1. Row Echelon Form
   a) \[
   \begin{pmatrix}
   1 & 2 & \mid & 1 \\
   0 & 3 & \mid & 2 \\
   \end{pmatrix}
   \]
   In REF, pivots are 1 and 3
   b) \[
   \begin{align*}
   5x - y &= 1 \\
   0x + 0y &= 6 \\
   0x + 0y &= 0
   \end{align*}
   \]
   In REF, pivots are 5 and 6
   c) \[
   \begin{pmatrix}
   2 & 1 & -1 & \mid & 1 \\
   0 & 3 & 1 & \mid & 2 \\
   2 & 0 & 0 & \mid & 1 \\
   \end{pmatrix}
   \]
   Not in REF
   d) \[
   \begin{align*}
   2x + y &= 3 \\
   y &= 5
   \end{align*}
   \]
   In REF, pivots are 2 and 1
   e) \[
   \begin{align*}
   3x + 2y + z &= 0 \\
   0x + 0y + 0z &= 0 \\
   -x - 2y + 4z &= 1
   \end{align*}
   \]
   Not in REF
   f) \[
   \begin{align*}
   5x + 5y &= 5 \\
   x + y &= 1
   \end{align*}
   \]
   Not in REF
2. Two Equations and Two Unknowns
   
   a) 
   
   b) The linear system is
   \[
   \begin{align*}
   x - y &= 2 \\
   2x - 4y &= -4.
   \end{align*}
   \]
   Subtract \(2 \cdot R_1\) from \(R_2\) to obtain:
   \[
   \begin{align*}
   x - y &= 2 \\
   -2y &= -8.
   \end{align*}
   \]
   
   c) Divide the second row by 2 to obtain:
   \[
   \begin{align*}
   x - y &= 2 \\
   y &= 4.
   \end{align*}
   \]
   
   d) Add the second row to the first row to obtain:
   \[
   \begin{align*}
   x &= 6 \\
   y &= 4.
   \end{align*}
   \]
   
   This is the solution.
   
   e) \(6 - 4 = 2, \ 2 \cdot 6 - 4 \cdot 4 = -5\).
   
   f) The system first becomes in REF after the 1st row operation. The pivots are 1 and \(-2\).
3. Three Equations Three Unknowns

a) \( A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}. \)

b) The augmented matrix is
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{pmatrix}.
\]

c) First, replace \( R_2 \) by \( R_2 - 2R_1 \) (\( R_2 += -2R_1 \)).
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{pmatrix}.
\]

Then \( R_3 += 6R_1 \):
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{pmatrix}.
\]

Now, you can do row scaling here, although you don't need to. Let's do it now to simplify our rows: \( R_2 \times= -(1/2) \) and \( R_3 \times= -(1/3) \) (combining two elementary operations at once):
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 5 & 3 & -11 \end{pmatrix}.
\]

We do one more row addition, replacing \( R_2 \) with \( R_2 - 5R_1 \) (\( R_2 -= 5R_1 \)):
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 18 & -36 \end{pmatrix}.
\]

Do one more row scaling, replacing \( R_3 \) with \( \frac{1}{18}R_3 \) (\( R_3 \times= 1/18 \)):
\[
\begin{pmatrix} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{pmatrix}.
\]

d) I used 6 elementary row operations, but the row scalings could have been avoided, giving you as few as 3.

e) The system of equations is now
\[
\begin{align*}
x_1 - 3x_2 + x_3 &= 4 \\
x_2 - 3x_3 &= 5 \\
x_3 &= -2
\end{align*}
\]
Substituting $x_3 = -2$, we obtain the system
\[
\begin{align*}
x_1 - 3x_2 &= 6 \\
x_2 &= -1 \\
x_3 &= -2
\end{align*}
\]
Substituting $x_2 = -1$, we obtain the system
\[
\begin{align*}
x_1 &= 3 \\
x_2 &= -1 , \\
x_3 &= -2
\end{align*}
\]
which is the solution.

\[f) \text{ Check } \begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}.\]

4. **Another One—What’s Different?**

Consider the system of three linear equations
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
2x_1 - 4x_2 + 8x_3 &= 2 \\
x_1 - 3x_2 - x_3 &= 1.
\end{align*}
\]

a) The linear system is
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
2x_1 - 4x_2 + 8x_3 &= 2 \\
x_1 - 3x_2 - x_3 &= 1.
\end{align*}
\]
By doing two row subtraction operations ($R_2 \leftarrow 2R_1$ and $R_3 \leftarrow R_1$), we obtain
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
6x_3 &= 6 \\
-x_2 - 2x_3 &= 3.
\end{align*}
\]

b) We swap rows 1 and 2 to obtain
\[
\begin{align*}
x_1 - 2x_2 + x_3 &= -2 \\
-x_2 - 2x_3 &= 3 \\
6x_3 &= 6.
\end{align*}
\]

c) Dividing row 3 by 6 gives $x_3 = 1$, which we substitute into the first two equations:
\[
\begin{align*}
x_1 - 2x_2 &= -3 \\
-x_2 &= 5 \\
x_3 &= 1.
\end{align*}
\]
Dividing row 2 by $-1$ gives $x_2 = -5$, which we substitute into the 1st equation:
\[
\begin{align*}
x_1 &= -13 \\
x_2 &= -5 \\
x_3 &= 1.
\end{align*}
\]
This is the solution.
5. Traffic Jam

a) We start with

\[
120 + w = 250 + x \\
120 + x = 70 + y \\
390 + y = 250 + z \\
115 + z = 175 + w
\]

or

\[
x - w = -130 \\
-x + y = 50 \\
-y + z = 140 \\
-z + w = -60.
\]

b) Eliminating \( x \) from the second equation gives

\[
x - w = -130 \\
y - w = -80 \\
-y + z = 140 \\
-z + w = -60.
\]

c) Eliminating \( y \) from the third equation gives

\[
x - w = -130 \\
y - w = -80 \\
z - w = 60 \\
-z + w = -60.
\]

d) Eliminating \( z \) from the fourth equation gives

\[
x - w = -130 \\
y - w = -80 \\
z - w = 60 \\
0 + 0 = 0.
\]

e) We can't just use substitution, as our final equation is not of the form \( w = (?) \).

The number of cars on roads \( x, y, \) and \( z \) all depend on how many cars are on \( w \).

f) The system has infinitely many solutions. There can be as many cars as you want, travelling in a circle around the town.

g) The augmented matrix is

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & -130 \\
0 & 1 & 0 & -1 & -80 \\
0 & 0 & 1 & -1 & 60 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The pivots are the 1’s. Not every row has a pivot. The fourth column does not have a pivot - as we will discuss in week 3, this means that we can find a solution which makes the fourth variable take any value we want. Such a variable is called a free variable.
6. Reduced Row Echelon Form

(1) No; Yes; Yes; Yes.
(2) No; No; No; No.

(3) (a) Use Gaussian elimination to obtain \[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(b) Use Jordan substitution to obtain \[
\begin{pmatrix}
1 & 0 & 0 & -4 \\
0 & 1 & 0 & 3/2 \\
0 & 0 & 1 & 2
\end{pmatrix}.
\]

(c) Use Jordan substitution to obtain \[
\begin{pmatrix}
1 & 0 & -1/2 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{pmatrix}.
\]

(d) Use Jordan substitution to obtain \[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

(4) (a) Pivots: 1, 1, 1. No solution.
(b) Pivots: 1, 2, 1. Unique solution.
(c) Pivots: 2, −1. Unique solution.
(d) Pivots: 1, 1. No solution.