1. **Matrices with complex eigenvalues**

Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) Compute the eigenvalues of $A$ and $B$. Write each eigenvalue in polar coordinates $z = re^{i\theta}$.

b) Compute the eigenvectors of $A$ and $B$. 

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### Math 218D Problem Session

**Week 11**

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2. **The dynamics of a diagonal matrix**

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

**a)** For each of the following vectors, plot $v$, $Av$, and $A^2v$:

1. $v = (1, 0)$
2. $v = (0, 1)$
3. $v = (1, 1)$

**b)** For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v, A^{-1}v, v, Av, A^2v, \ldots$.

**c)** For the vector $v = (1, 1)$, what direction is the vector $A^n v$ approximately pointing when $n$ is very large? In other words, what unit vector does $\frac{A^n v}{\|A^n v\|}$ approximate when $n$ is very large?

**d)** For the vector $v = (1, 1)$, what direction is $A^{-n} v$ approximately pointing when $n$ is very large?
3. **The dynamics of a diagonalizable matrix**
Consider the matrix $A$ with $A(1, 1) = 3(1, 1)$ and $A(1, -2) = 2(1, -2)$. In other words, $A$ is diagonalizable and you have been told the eigenvectors and eigenvalues.

a) For each of the following vectors, plot $v$, $Av$, $A^2v$:
   (1) $v = (1, 1)$
   (2) $v = (1, -2)$
   (3) $v = (2, -1)$

   *You can do this without computing the matrix $A$!*

b) For each of the same vectors, sketch the shape you get by connecting the dots between the points ..., $A^{-2}v$, $A^{-1}v$, $v$, $Av$, $A^2v$, ...

c) For the vector $v = (2, -1)$, what direction is the vector $A^n v$ approximately pointing when $n$ is very large?

d) For the vector $v = (2, -1)$, what direction is $A^{-n}v$ approximately pointing when $n$ is very large?
4. Dynamics of complex matrices

Consider the matrices

\[ A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix} \] and \[ B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]
you studied in problem 1.

a) Plot the points \((4, 0), A(4, 0), A^2(4, 0), A^3(4, 0), A^4(4, 0)\). Connect the dots between these points. Predict the shape that you would get if you continued to \(A^5(4, 0), A^6(4, 0), \ldots\).

b) Plot the points \((1, 0), B(1, 0), B^2(1, 0), B^3(1, 0), B^4(1, 0)\). Connect the dots between these points. Predict the shape that you would get if you continued to \(B^5(4, 0), B^6(4, 0), \ldots\).

c) What do the eigenvalues you found in problem 1a) explain about your pictures from a) and b)?

d) Find complex scalars \(a, b\) such that \((1, 0) = av_1 + bv_2\), where \(v_1\) and \(v_2\) are the eigenvectors for \(B\) you found in problem 1c).

e) Compute \(B^n(1, 0)\) in terms of complex exponentials.

f) Use Euler's formula \(e^{i\theta} = \cos(\theta) + i\sin(\theta)\) to write \(B^n(1, 0)\) in terms of trig. functions (no complex numbers should appear in your final answer).

g) Can you predict a formula for \(A^n(4, 0)\) in terms of trig. functions?
5. Some quick matrix exponentials

Compute the matrix exponential $e^A$ of:

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$,

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,

(3) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. 
6. **A differential equation**

Consider the system of differential equations

\[
\begin{align*}
x'(t) &= 3x(t) + 2y(t) \\
y'(t) &= 4x(t) - 4y(t)
\end{align*}
\]

a) Write this as a matrix differential equation

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]

What is the matrix \( A \)?

b) For this matrix \( A \), find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), as well as the eigenvectors \( w_1 \) and \( w_2 \).

c) Every solution is of the form \((x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2\). If you want the solution to have initial value \((x(0), y(0)) = (1, 1)\), which scalars \( a_1 \) and \( a_2 \) should you choose?

d) Plug the solution with initial value \((x(0), y(0)) = (1, 1)\) to the differential equation, and confirm that it is a solution.

e) For the solution you found in c), compute \((x(1), y(1))\).
7. **A complex ODE**

Consider the system of differential equations

\[
\begin{align*}
    x'(t) &= x(t) - y(t), \\
    y'(t) &= x(t) + y(t).
\end{align*}
\]

a) Write this as a matrix differential equation \( \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \).

b) Compute the eigenvalues \( \lambda_1, \lambda_2 \) and eigenvectors \( v_1, v_2 \) of the matrix \( A \).

c) Compute the real and imaginary parts of the “eigenvector solution” \( (x(t), y(t)) = e^{\lambda_1 t} v_1 \). This gives you two different real solutions to the differential equation.

d) Find the solution \( (x(t), y(t)) \) with initial value \( (x(0), y(0)) = (1, 0) \).