Subspaces

Orientation: We're developing machinery to "almost solve" $Ax=b$

So far, to every matrix $A$ we have associated two spans:

1. the span of the columns/all $b$ such that $Ax=b$ is consistent
2. the solution set of $Ax=0$

The first arises naturally as a span—it is already in parametric form. The second required work (elimination) to write as a span—it is a solution set, so it is in implicit form.

The notion of subspaces puts both on the same footing. This formalizes what we mean by "linear space containing $0$".

Fast-forward: Subspaces are spans and Spans are subspaces.

Why the new vocabulary word?
When you say "span" you have a spanning set of vectors in mind (parametric form). This is not the case for the solutions of $Ax=0$. 
Subspaces allow us to discuss spans without computing a spanning set. \( \text{Subspace} = \text{Span} \{ \text{???} \} \)

They also give a criterion for a subset to be a span.

**Def:** A subset of \( \mathbb{R}^n \) is any collection of points.

**Eg:**

(a) \( \{(x, y) : x^2 + y^2 = 1\} \)

(b) \( \{(x, y) : x, y \geq 0\} \)

(c) \( \{(x, y) : xy = 0\} \)

**Def:** A subspace is a subset \( V \) of \( \mathbb{R}^n \) satisfying:

1. [closed under +] If \( u, v \in V \) then \( u + v \in V \)
2. [closed under scalar x] If \( u \in V \) and \( c \in \mathbb{R} \) then \( cu \in V \)
3. [contains 0] \( 0 \in V \)

These conditions characterize linear spaces containing 0 among all subsets.

**NB:** If \( V \) is a subspace and \( v \in V \) then \( 0 = 0v \) is in \( V \) by (2), so (3) just means \( V \) is nonempty.
In the subsets above:

(a) fails (1), (2), (3)
(b) fails (2): \((1) \in V\) but \(-1 \cdot (1) \notin V\)
(c) fails (1): \((0), (0) \in V\) but \((1) \notin V\)

Here are two "trivial" examples of subspaces:

\[ \{0\} \text{ is a subspace} \]

1. \(0 + 0 = 0 \in \{0\}\) \(\checkmark\)
2. \(c \cdot 0 = 0 \in \{0\}\) \(\checkmark\)
3. \(0 \in \{0\}\)

NB \(\{0\} = \text{Span}\{0\}\): it is a span

\[ \mathbb{R}^n = \text{all vectors of size } n \text{ is a subspace} \]

1. The sum of two vectors is a vector. \(\checkmark\)
2. A scalar times a vector is a vector. \(\checkmark\)
3. \(0\) is a vector.

NB \(\mathbb{R}^n = \text{Span}\{e_1, e_2, \ldots, e_n\}\)

\[ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots, \quad e_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
defining condition

\[ V = \{ (x, y, z) : x+y = z \} \]

The defining condition tells you if \((x, y, z)\) is in \(V\) or not.

1. We have to show that if \((x_1, y_1, z_1) \in V\) and \((x_2, y_2, z_2) \in V\) then their sum is in \(V\). That means it also satisfies the defining condition.

\[
\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}
\]

Is \(z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\)? Yes, because \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) satisfy the defining condition: \(x_1 + y_1 = z_1\), \(x_2 + y_2 = z_2\).

2. We have to show that if \((x, y, z) \in V\) and \(c \in \mathbb{R}\) then \(c(x, y, z) \in V\).

\[
c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \]

Is \(cx + cy = cz\)?

Yes, because \(x+y = z\).

3. Is \(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V\)? Does it satisfy the defining condition?

\(0 + 0 = 0\)

Since \(V\) satisfies the 3 criteria, it is a subspace.
Defining condition

Eg: \( V = \{(x,y) : x \geq 0, y \geq 0\} \)

1. We have to show that if \((x_1,y_1) \in V\) and \((x_2,y_2) \in V\) then \((x_1+x_2,y_1+y_2) \in V\).
   - Is \(x_1+x_2 \geq 0\)? Yes, because \(x_1 \geq 0, x_2 \geq 0\)
   - Is \(y_1+y_2 \geq 0\)? Yes, because \(y_1 \geq 0, y_2 \geq 0\).

2. We have to show that if \((x,y) \in V\) and \(c \in \mathbb{R}\) then \((cx, cy) \in V\).
   - Is \(cx \geq 0\)? Not necessarily!
     - Fails if \(c < 0, x > 0\).

\[ V \]

**Good:** this is not a picture of a span.

In practice you will rarely check that a subset is a subspace by verifying the axioms.

**Fact:** A span is a subspace

**Proof:** Let \( V = \text{Span} \{ v_1, \ldots, v_n \} \).

Here the defining condition for a vector to be in \( V \) is that it is a linear combination of \( v_1, \ldots, v_n \).
(1) We need to show that if 
\( c_1v_1 + \ldots + c_nv_n \in V \) \& \( d_1v_1 + \ldots + d_nv_n \in V \)
then their sum is in \( V \): the sum of two linear combos of \( v_1, \ldots, v_n \) is a linear combo.
\[
(c_1v_1 + \ldots + c_nv_n) + (d_1v_1 + \ldots + d_nv_n) = (c_1d_1)v_1 + \ldots + (c_n+d_n)v_n \in V \] \checkmark

(2) We need to show that if \( c_1v_1 + \ldots + c_nv_n \in V \) and \( d \in \mathbb{R} \) then the product is in \( V \).
\[
d(c_1v_1 + \ldots + c_nv_n) = (dc_1)v_1 + \ldots + (dc_n)v_n \in V \] \checkmark

(3) Every span contains \( 0 \):
\[
0 = 0v_1 + \ldots + 0v_n \] \checkmark

Conversely, suppose \( V \) is a subspace.
If \( v_1, \ldots, v_n \in V \) and \( c_1, \ldots, c_n \in \mathbb{R} \) then:
\[
c_1v_1 + \ldots + c_nv_n \in V \quad \text{by (2)}
\]
\[
c_1v_1 + c_2v_2 \in V \quad \text{by (1)}
\]
\[
(c_1v_1 + c_2v_2) + c_3v_3 \in V \quad \text{by (1)}
\]
\[
\vdots
\]
\[
c_1v_1 + \ldots + c_nv_n \in V
\]
so \( \text{Span} \{v_1, \ldots, v_n\} \) is contained in \( V \).
Choose enough \( v_i \)'s to fill up \( V \), and you get:
Def: The column space of a matrix $A$ is the span of its columns.

Notation: $\text{Col}(A) = \text{Span}\text{ \{\text{cols of } A\}}$

This is a subspace of $\mathbb{R}^m$, $m = \#\text{rows}$

(each column has $m$ entries)

$\sim$ column picture.

Since a column space is a span & a span is a subspace, a column space is a subspace.

Eg: $\text{Col}\begin{bmatrix} 1/3 & 4/6 & 7/9 \end{bmatrix} = \text{Span}\begin{bmatrix} 1/3 \end{bmatrix}$

It's easy to translate between spans & column spaces.

Eg: $\text{Span}\begin{bmatrix} 1/3 \end{bmatrix}$ $\sim$ $\text{Col}\begin{bmatrix} 1/3 \end{bmatrix}$

NB: $\text{Col}(A) = \{Ax: x \in \mathbb{R}^n\}$

because "Ax" is just a linear combination of the cols of $A$. 
Translation of the column picture criterion for consistency:

\[ \text{Ax=b is consistent } \iff \text{b} \in \text{Col}(A) \]

(\text{this is just substituting \textit{Col}(A) for \textit{the span of the columns of } A)\]

\textbf{Def:} The \textit{null space} of a matrix \( A \) is the solution set of \( Ax=0 \).

\textbf{Notation:} \( \text{Nul}(A) = \{ x \in \mathbb{R}^n : Ax = 0 \} \)

This is a subspace of \( \mathbb{R}^n \) \( n = \# \text{ columns} \)
\( (n = \# \text{variables and } \text{Nul}(A) \text{ is a solution set}) \)

\textbf{Fact:} \( \text{Nul}(A) \) is a subspace

Of course we also know \( \text{Nul}(A) \) is a span, but we can verify this directly.

\textbf{Proof:} The defining condition for \( \text{v} \in \text{Nul}(A) \) is that \( Av = 0 \).

(1) Say \( u, v \in \text{Nul}(A) \). Is \( u + v \in \text{Nul}(A) \)?
\[ A(u + v) = Au + Av = 0 + 0 = 0 \]
(2) Say \( u \in \text{Null}(A) \) and \( c \in \mathbb{R} \). Is \( cu \in \text{Null}(A) \)?
\[
A(cu) = c(Au) = c \cdot 0 = 0
\]
(3) Is \( 0 \in \text{Null}(A) \)?
\[
A0 = 0
\]
This is an example of a subspace that does not come with a spanning set!
⇒ It's much more natural to consider it as a subspace when reasoning about it.

How to produce a spanning set for a null space?

\[ \text{Null}(A) \quad \text{span} \{ \ldots \} \]

**Parametric vector form**

\[ \text{(Gauss-Jordan eliminations)} \]

\[ \text{Work} \]

**Eg**: Write \( \text{Null}(A) \) as a span for
\[
A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}
\]
This means solving \( Ax = 0 \) (homogeneous equation).
\[
\begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

Parametric form:
\[
\begin{align*}
X_1 &= -2X_2 + X_4 \\
X_2 &= X_2 \\
X_3 &= -X_4 \\
X_4 &= X_4
\end{align*}
\]

\[
\begin{bmatrix}X_1 \\ X_2 \\ X_3 \\ X_4\end{bmatrix} \xrightarrow{\text{PVE}} X_2 \begin{bmatrix}-2 \\ 1 \\ 0 \\ 0\end{bmatrix} + X_4 \begin{bmatrix}1 \\ 0 \\ 0 \\ 1\end{bmatrix}
\]

\[
\Rightarrow \text{Null}(A) = \text{Span} \left\{ \begin{bmatrix}-2 \\ 1 \\ 0 \\ 0\end{bmatrix}, \begin{bmatrix}1 \\ 0 \\ 0 \\ 1\end{bmatrix} \right\}
\]

**NB:** Any two non-collinear vectors span a plane, so Null(A) will have many different spanning sets.

eg \[
\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix}1 \\ 0 \\ 0 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ 1 \\ 0 \\ 0\end{bmatrix} \right\}
\]

More on this later.

**NB:** Likewise for the column space: eg.
\[
\text{Col} \left( \begin{bmatrix}1 \\ 0 \\ 0 \end{bmatrix} \right) = \text{Col} \left( \begin{bmatrix}1 \\ 0 \\ 1 \end{bmatrix} \right) = \text{Col} \left( \begin{bmatrix}1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \text{(xy-plane)}
\]
Implicit vs Parametric form:

- **Col(A)** is a span:
  \[ \text{Col}(A) = \{ x_{v_1} + \cdots + x_{v_m} : x_{y_1} \ldots x_{y_n} \in \mathbb{R} \} \]
  where \( v_1, \ldots, v_m \) are the columns of \( A \).

  \[ \Rightarrow \text{parametric form} \]

- **Null(A)** is a solution set:
  \[ \text{Null} \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix} \]
  \[ = \{ (x_1, x_2, x_3, x_4) : x_1 + 2x_2 + 2x_3 + x_4 = 0, 2x_1 + 4x_2 + x_3 - x_4 = 0 \} \]

  \[ \Rightarrow \text{implicit form} \]

In practice you will (almost) always write a subspace as a column space/\( \text{span} \) or a null space. Which one?

- **parameters?** \( \Rightarrow \text{Col}(A) / \text{Span} \)
- **equations?** \( \Rightarrow \text{Null}(A) \)

Once you're done this, you can ask a computer to do computations on it!
Eg: \( V = \{ (x, y, z) : x + ty = z \} \)
This is defined by the equation \( x + ty = z \).
rewrite: \( x + y - z = 0 \)
\( \therefore V = \text{Null} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \)

Eg: \( V = \{ \left( \frac{3a+b}{a-b} \right) : a, b \in \mathbb{R} \} \)
This is described by parameters. Rewrite:
\( \left( \frac{3a+b}{a-b} \right) = a (\frac{3}{0}) + b (-1) \)
\( \therefore V = \text{Span} \{ (3, 0), (-1) \} = \text{Col} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \)

This is also how you should verify that a subset is a subspace.

Of course, if \( V \) is not a subspace then you can’t write it as \( \text{Col}(A) \) or \( \text{Null}(A) \). In this case you should check that it fails one of the axioms.

Eg: Is \( V = \{ (x, y, z) : x + ty = z + 1 \} \) a subspace?
No, (P3) fails: \( 0 + 0 \neq 0 + 1 \), so \( 0 \notin V \).