Eg: Diagonalize $A = (0 \; -1) \quad (\text{CCW rotation by } 90^\circ)$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - tr(A)\lambda + \det(A) = \lambda^2 + 1$$

This has no real roots: $\lambda^2 + 1 = 0 \iff \lambda^2 = -1$.

Solution: Add a $\sqrt{-1}$ to our number system!

Def: The unit imaginary number is a number $i$ such that $i^2 = -1$. A complex number is a number $a+bi$ for $a, b \in \mathbb{R}$.

$$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$$

is the set of all complex numbers.

If $z = a+bi$ is a complex number, its

- real part is $\text{Re}(z) = a$, and its
- imaginary part is $\text{Im}(z) = b$.

We can add/subtract & multiply complex numbers:

$$(a+bi) \pm (c+di) = (a\pm c) + (b\pm d)i$$

$$(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

Division: see p.4
Question: Wait! Why can I just declare that -1 has a square root?

Answer 1: Why can you declare that 2 has a square root? You can't write it down—it's an infinite non-repeating decimal...


Complex numbers have an additional algebraic operation.

Def: The complex conjugate of $z = a + bi$ is

$$\bar{z} = a - bi$$

(replace $i$ by $-i$ = the other $\sqrt{-1}$)

Check: $\sqrt{z + \bar{z}} = z + \bar{z}$

$z \cdot \bar{z} = z \cdot \bar{z}$

$\bar{z}^2 = z$

NB: if $z = a + bi$ then

$z + \bar{z} = (a + bi) + (a - bi) = 2a = 2 \text{Re}(z)$

$z - \bar{z} = (a + bi) - (a - bi) = 2bi = 2i \text{Im}(z)$

$z + \bar{z} = 2 \text{Re}(z)$

$z - \bar{z} = 2i \text{Im}(z)$
Since a complex number \( z = a + bi \) is determined by two real numbers \( a \) and \( b \), we can draw \( C \) as a plane:

\[
\text{The Complex Plane}
\
\begin{array}{c}
\text{Imaginary Axis} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Complex Conjugation} \\
\end{array}
\]

Complex conjugation negates the imaginary coordinate: it flips over the real axis.

**NB:** A real number is also a complex number: 
\( a \in \mathbb{R} \Rightarrow a + 0i \in \mathbb{C} \)

So \( \mathbb{C} \) contains \( \mathbb{R} \). **NB:** \( z \in \mathbb{R} \Leftrightarrow z = \overline{z} \)

**NB:** If \( z = a + bi \) then
\[
\overline{z} = (a + bi)(a - bi) = a^2 - bi^2 = a^2 + b^2 \Rightarrow \text{nonnegative real number}
\]

\[
\boxed{z = a + bi \Rightarrow \overline{z} = a^2 + b^2 \geq 0}
\]

**Def:** The modulus of \( z \) is \( |z| = \sqrt{\overline{z}z} \)

This is its length as a vector in the complex plane.
Eg: \( z = 2 + i \)
\[ \Rightarrow |z| = \sqrt{4+1} = \sqrt{5} \]

Eg: If \( a \in \mathbb{R} \) then \( a = a + 0i \) and

\[ |a| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a| \]
(the modulus is the usual absolute value)

Check: \( |zw| = |z| \cdot |w| \quad |\bar{z}| = |z| \)

Here’s how to take a reciprocal of \( z = a + bi \neq 0 \):

\[ \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{zz} = \frac{\bar{z}}{|z|^2} \]

\[ \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{or} \quad \frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \]

Check: \( \left| \frac{1}{z} \right| = \frac{1}{|z|} \)

Eg: \( \frac{1}{5 + i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5} i \)

Check: \( (2+i)(\frac{2}{5} - \frac{1}{5} i) = \frac{4}{5} + \frac{1}{5} + (\frac{2}{5} - \frac{2}{5}) i = 1 \checkmark \)

Now we can divide too: \( \frac{\bar{z}}{z} = \omega \cdot \frac{1}{z} = \frac{\bar{z}}{|z|^2} \).
Polar Coordinates

Recall that a point in the \((x,y)\)-plane can be specified in polar coordinates \((r,\theta)\):

- \(r = \text{length of } (x,y) = \sqrt{x^2 + y^2}\)
- \(\theta = \text{angle } (x,y) \text{ makes with the positive } x\text{-axis } = \pm \arctan\left(\frac{y}{x}\right)\)

To go from polar coordinates back to Cartesian \((x,y)\)-coordinates:

\[
(r,\theta) \rightarrow x = r\cos\theta \\
y = r\sin\theta
\]

If we apply this to a complex number \(z = a + bi\):

\[
r = \sqrt{a^2 + b^2} = |z| \\
\rightarrow a = |z|\cos\theta \\
b = |z|\sin\theta
\]

\[
\Rightarrow z = |z| (\cos\theta + i\sin\theta)
\]

Def: The argument of \(z = a + bi\) is

\[
\arg(z) = \theta = \text{the angle } z \text{ makes with the positive real axis.}
\]
So you can specify a complex number in 2 ways:

(Cartesian coords / real & imaginary parts)

\[ z = a + bi \]

(Polar coords / modulus & argument)

\[ z = |z| (\cos \Theta + i \sin \Theta) \quad \Theta = \arg(z) \]

**Eg:** \( z = 1 + i \rightarrow |z| = \sqrt{1 + 1} = \sqrt{2} \)

\[ \arg(z) = 45^\circ \]

\[ \Rightarrow z = \sqrt{2} (\cos(45^\circ) + i \sin(45^\circ)) \]

**Check:** \( \cos(45^\circ) = \sin(45^\circ) = \frac{1}{\sqrt{2}} \)

\[ z = 1 + i = \sqrt{2} (\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}) \checkmark \]

**Facts:**

- \( \arg(\overline{z}) = -\arg(z) \) (flip over real axis)

- \( \arg(1/z) = -\arg(z) \)

\( \left( \frac{1}{z} = \frac{1}{|z|^2} = (\text{positive number}) \cdot \overline{z} \right) \)
Fact: $\arg(zw) = \arg(z) + \arg(w)$

Proof: This is a trig identity!

$z = |z| (\cos \theta + i \sin \theta) \quad \theta = \arg(z)$

$w = |w| (\cos \phi + i \sin \phi) \quad \phi = \arg(w)$

$zw = |zw| (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)$

$= |zw| \left( \cos \theta \cos \phi - \sin \theta \sin \phi + (\cos \theta \sin \phi + \sin \theta \cos \phi)i \right)$

$= |zw| \left( \cos (\theta + \phi) + i \sin (\theta + \phi) \right)$

$\Rightarrow \theta + \phi = \arg(zw)$

We like polar coordinates because multiplication is easier!

Exercise: expand the product

$(\cos \theta + i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta)$

to derive the triple-angle formulas.
Euler's Formula: For any real number $\theta$,

$$e^{i\theta} = \cos \theta + i\sin \theta$$

Proof: Take Taylor expansions of both sides...

Polar Form of Complex Numbers, Alternative:

$$z = |z|e^{i\theta} \quad \theta = \arg(z)$$

This makes the formulas on p.7 easier to remember:

$$zw = (|z|e^{i\theta})(|w|e^{i\phi}) = |z||w|e^{i(\theta+\phi)}$$

$$\bar{z} = |z|e^{-i\theta}$$

$$\frac{1}{z} = (|z|e^{i\theta})^{-1} = \frac{1}{|z|}e^{-i\theta}$$

Eg: $-1 = e^{i\pi}$ \quad $1+i = \sqrt{2}e^{i\pi/4}$ (cf. p.6)

You can exponentiate any complex number:

$$e^{a+bi} = e^a e^{bi} = e^a (\cos b + i\sin b)$$

Since $\sin(-b) = -\sin b$, we have

$$e^z = \overline{e^z} \quad \text{for any } z \in \mathbb{C}$$
It turns out that once $x^2+1=0$ has a solution, then any polynomial has a root!

**Fundamental Theorem of Algebra:**

Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

$$a_0, a_1, \ldots, a_n \in \mathbb{C} \quad a_n \neq 0$$

can be factored into linear terms:

$$p(x) = a_n (x - \lambda_1) \cdots (x - \lambda_n) \quad \lambda_1, \ldots, \lambda_n \in \mathbb{C}$$

**Eg:** $p(x) = x^2 + x + 1$

Use the quadratic formula:

$$x = \frac{1}{2} \left( -1 \pm \sqrt{1 - 4} \right) = \frac{1}{2} \left( -1 \pm i\sqrt{3} \right)$$

$$\sqrt{-3} = \sqrt{-1} \cdot \sqrt{3} = i\sqrt{3}$$

So, $p(x) = (x - \frac{1}{2}(-1 + i\sqrt{3}))(x - \frac{1}{2}(-1 - i\sqrt{3}))$

**Eg:** $p(x) = x^2 + 1 = (x + i)(x - i)$

So now $(0, -1)$ has two eigenvalues $\pm i$ so it's diagonalizable!
Real Polynomials:

If \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) has real coefficients \( a_0, a_1, \ldots, a_n \in \mathbb{R} \), then its complex roots come in conjugate pairs:

\[ p(x) = 0 \iff p(\bar{x}) = 0 \]

Check:

\[
p(\bar{x}) = a_n \bar{x}^n + a_{n-1} \bar{x}^{n-1} + \cdots + a_1 \bar{x} + a_0
\]

\[
(\bar{a}_i = a_i) = a_n \bar{x}^n + a_{n-1} \bar{x}^{n-1} + \cdots + a_1 \bar{x} + a_0 = p(\bar{x})
\]

So \( p(x) = 0 \iff p(\bar{x}) = 0 \).

Eg: The roots of \( p(x) = x^2 + x + 1 \) are

\[
\lambda = \frac{1}{2}(-1 + i\sqrt{3}) \quad \text{and} \quad \bar{\lambda} = \frac{1}{2}(-1 - i\sqrt{3})
\]