Homework #12
due Wednesday, April 6, at 11:59pm

1. For each symmetric matrix \( S \), find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( S = QDQ^T \).
   
   a) \[
   \begin{pmatrix}
   1 & -3 \\
   -3 & 1
   \end{pmatrix}
   \]
   b) \[
   \begin{pmatrix}
   1 & -3 \\
   -3 & 9
   \end{pmatrix}
   \]
   c) \[
   \begin{pmatrix}
   14 & 2 \\
   2 & 11
   \end{pmatrix}
   \]
   d) \[
   \begin{pmatrix}
   7 & 2 & 0 \\
   2 & 6 & 2 \\
   0 & 2 & 5
   \end{pmatrix}
   \]
   e) \[
   \begin{pmatrix}
   1 & -8 & 4 \\
   -8 & 1 & 4 \\
   4 & 4 & 7
   \end{pmatrix}
   \]

   The eigenvalues in d) are 3, 6, 9 and in e) are \(-9, 9\).

2. For each matrix \( S \) of Problem 1, decide if \( S \) is positive-semidefinite, and if so, compute its positive-semidefinite square root \( \sqrt{S} = Q \sqrt{D} Q^T \). Verify that \((\sqrt{S})^2 = S\).

   Remark: Since \( \sqrt{S} \) is also symmetric, we have \( S = \sqrt{S}^T \sqrt{S} \), so this is another way to factorize a positive-semidefinite matrix as \( A^T A \).

3. Consider the matrix
   \[
   S = \begin{pmatrix}
   7 & 2 & 0 \\
   2 & 6 & 2 \\
   0 & 2 & 5
   \end{pmatrix}
   \]

   of Problem 1(d). Write \( S \) in the form \( \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T \) for numbers \( \lambda_1, \lambda_2, \lambda_3 \) and orthonormal vectors \( u_1, u_2, u_3 \).

4. Find all possible orthogonal diagonalizations
   \[
   \frac{1}{5} \begin{pmatrix}
   41 & 12 \\
   12 & 34
   \end{pmatrix} = QDQ^T.
   \]

5. Let \( S \) be a symmetric matrix such that \( S^k = 0 \) for some \( k > 0 \). Show that \( S = 0 \).
   [Hint: Use HW10#17.]

6. Let \( S \) be a symmetric orthogonal \( 2 \times 2 \) matrix.
   a) Show that \( S = \pm I_2 \) if it has only one eigenvalue.
      [Hint: See HW10#11.]
   b) Suppose that \( S \) has two eigenvalues. Show that \( S \) is the matrix for the reflection over a line \( L \) in \( \mathbb{R}^2 \). (Recall that the reflection over a line \( L \) is given by \( R_L = I_2 - 2P_{L^\perp} \).)
      [Hint: Write \( S \) as \( \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T \), and use the projection formula to write \( I_2 \) and \( P_{L^\perp} \) in this form as well. What is \( L \)?]
7. a) Let $S$ be a diagonalizable (over $\mathbb{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that $S$ is symmetric.

[Hint: choose orthonormal bases for each eigenspace.]

b) Let $S$ be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that $S$ is symmetric.

c) Let $V$ be a subspace of $\mathbb{R}^n$, and let $P_V$ be the projection matrix onto $V$. Use a) or b) to prove that $P_V$ is symmetric. (We proved this in class using the formula $P_V = A(A^T A)^{-1} A^T$.)

8. For which matrices $A$ is $S = A^T A$ positive-definite? If $S$ is not positive-definite, find a vector $x$ such that $x^T S x = 0$. In any case, do not compute $S$!

a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}$  

b) $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \\ 7 & 8 & 9 \end{pmatrix}$  

c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

9. a) If $S$ is positive-definite and $C$ is invertible, show that $CSC^T$ is positive-definite.

b) If $S$ and $T$ are positive-definite, show that $S + T$ is positive-definite.

c) If $S$ is positive-definite, show that $S$ is invertible and that $S^{-1}$ is positive-definite.

[Hint: For a) and b) use the positive-energy characterization of positive-definiteness; for c) use the positive-eigenvalue characterization.]

10. Let $S$ be a positive-definite matrix.

a) Show that the diagonal entries of $S$ are positive.

[Hint: compute $e_i^T S e_i$.]

b) Show that the diagonal entries of $S$ are all greater than or equal to the smallest eigenvalue of $S$.

[Hint: if not, apply a) to $S - aI_n$ for a diagonal entry $a$ that is smaller than all eigenvalues.]

11. Decide if each statement is true or false, and explain why. All matrices are real.

a) A symmetric matrix is diagonalizable.

b) If $A$ is any matrix then $A^T A$ is positive-semidefinite.

c) A symmetric matrix with positive determinant is positive-definite.

d) If $A = CDC^{-1}$ for a diagonal matrix $D$ and a non-orthogonal invertible matrix $C$, then $A$ is not symmetric.
A positive-definite matrix has the form $A^T A$ for a matrix $A$ with full column rank.

The only positive-definite projection matrix is the identity.

All eigenvalues of a positive-definite symmetric matrix are positive real numbers.

12. For each symmetric matrix $S$, decide if $S$ is positive-definite. If so, find its $LDL^T$ and Cholesky decompositions. Do not compute any eigenvalues!

   a) $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$  
   b) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix}$  
   c) $\begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$

   d) $\begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix}$  
   e) $\begin{pmatrix} -1 & 2 & 3 & -2 \\ 2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$

13. Consider the matrix

   $$S = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$  

   Without multiplying the matrices, find:

   a) The determinant of $S$.
   
   b) The eigenvalues of $S$.
   
   c) The eigenvectors of $S$.
   
   d) A reason why $S$ is symmetric positive-definite.

14. a) For each symmetric matrix $S$, compute the associated quadratic form $q(x) = x^T S x$.

   $$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

   b) Let $A$ be a square matrix and let $S = \frac{1}{2}(A + A^T)$. Show that $S$ is symmetric and that $x^T A x = x^T S x$. (This is why we only consider symmetric matrices when studying quadratic forms.)
15. For each quadratic form \(q(x_1, x_2)\), i) write \(q(x)\) in the form \(x^T S x\) for a symmetric matrix \(S\), ii) find coordinates \(y_1, y_2\) such that \(q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2\), and iii) find the maximum and minimum values of \(q(x_1, x_2)\) subject to the constraint \(x_1^2 + x_2^2 = 1\), and at which points \((x_1, x_2)\) these values are attained.

a) \(q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2\)
b) \(q(x_1, x_2) = \frac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2)\)
c) \(q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2\)

16. For the quadratic form \(q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3\), find coordinates \(y_1, y_2, y_3\) such that \(q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2\), and find the maximum and minimum values of \(x_1^2 + x_2^2 + x_3^2\) subject to the constraint \(q(x_1, x_2, x_3) = 1\), along with the points \((x_1, x_2, x_3)\) at which these values are attained.

17. Consider the quadratic form \(q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3\).
Find all vectors \(x = (x_1, x_2, x_3)\) maximizing \(q(x)\) subject to \(\|x\| = 1\). (There are infinitely many!)