Homework #12
Answer Key

1. For each symmetric matrix $S$, find an orthogonal matrix $Q$ and a diagonal matrix $D$ such that $S = QDQ^T$.
   
   a) $\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$  
   b) $\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$  
   c) $\begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$  
   d) $\begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$  
   e) $\begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$

   The eigenvalues in d) are 3, 6, 9 and in e) are $-9, 9$.

Solution.
   
   a) $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$  
      $D = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$
   
   b) $Q = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$  
      $D = \begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$
   
   c) $Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$  
      $D = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix}$
   
   d) $Q = \frac{1}{3} \begin{pmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$  
      $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$
   
   e) $Q = \begin{pmatrix} -1/\sqrt{2} & 1/3 \sqrt{2} & -2/3 \\ 1/\sqrt{2} & 1/3 \sqrt{2} & -2/3 \\ 0 & 4/3 \sqrt{2} & 1/3 \end{pmatrix}$  
      $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{pmatrix}$

2. For each matrix $S$ of Problem 1, decide if $S$ is positive-semidefinite, and if so, compute its positive-semidefinite square root $\sqrt{S} = Q\sqrt{D}Q^T$. Verify that $(\sqrt{S})^2 = S$.

Remark: Since $\sqrt{S}$ is also symmetric, we have $S = \sqrt{S}^T \sqrt{S}$, so this is another way to factorize a positive-semidefinite matrix as $A^T A$.

Solution.
   
   a) This matrix is indefinite.
   
   b) This matrix is positive-semidefinite, and
      $\sqrt{S} = \frac{1}{10} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{10} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$.
   
   c) This matrix is positive-definite, and
      $\sqrt{S} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{15} & 0 \\ 0 & \sqrt{10} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 4\sqrt{3} + \sqrt{2} & 2\sqrt{3} - 2\sqrt{2} \\ 2\sqrt{3} - 2\sqrt{2} & \sqrt{3} + 4\sqrt{2} \end{pmatrix}.$
d) This matrix is positive-definite, and
\[
\sqrt{S} = \frac{1}{9} \begin{pmatrix} 2 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{6} & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}
\]
\[
= \frac{1}{9} \begin{pmatrix} \sqrt{3} + 4\sqrt{6} + 12 & -2\sqrt{3} - 2\sqrt{6} + 12 & 2\sqrt{3} - 4\sqrt{6} + 6 \\ -2\sqrt{3} - 2\sqrt{6} + 12 & 4\sqrt{3} + \sqrt{6} + 12 & -4\sqrt{3} + 2\sqrt{6} + 6 \\ 2\sqrt{3} - 4\sqrt{6} + 6 & -4\sqrt{3} + 2\sqrt{6} + 6 & 4\sqrt{3} + 4\sqrt{6} + 3 \end{pmatrix}
\]
e) This matrix is indefinite.

3. Consider the matrix
\[
S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}
\]
of Problem 1(d). Write \(S\) in the form \(\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T\) for numbers \(\lambda_1, \lambda_2, \lambda_3\) and orthonormal vectors \(u_1, u_2, u_3\).

Solution.
\[
S = 3u_1 u_1^T + 6u_2 u_2^T + 9u_3 u_3^T \quad \text{for} \quad u_1 = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \quad u_2 = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \quad u_3 = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.
\]

4. Find all possible orthogonal diagonalizations
\[
\frac{1}{5} \begin{pmatrix} 41 & 12 & 12 \\ 12 & 34 & 12 \\ 12 & 12 & 34 \end{pmatrix} = QDQ^T.
\]

Solution.
The eigenvalues are \(\lambda_1 = 5\) and \(\lambda_2 = 10\) with eigenspaces spanned by \(w_1 = \frac{1}{5} \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}\) and \(w_2 = \frac{1}{5} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}\), respectively. Each eigenspace contains exactly two unit vectors, so the possible diagonalizations are:
\[
Q = \frac{1}{5} \begin{pmatrix} \pm 3 & \pm 4 & \pm 3 \\ \pm 4 & \pm 3 & \pm 4 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}.
\]

(There are eight in total.)

5. Let \(S\) be a symmetric matrix such that \(S^k = 0\) for some \(k > 0\). Show that \(S = 0\).

[Hint: Use HW10#17.]

Solution.
If $S$ is symmetric then it is diagonalizable. All eigenvalues are zero, so $S = QOQ^T = 0$.

6. Let $S$ be a symmetric orthogonal $2 \times 2$ matrix.
   a) Show that $S = \pm I_2$ if it has only one eigenvalue.
      [Hint: See HW10#11.]
   b) Suppose that $S$ has two eigenvalues. Show that $S$ is the matrix for the reflection over a line $L$ in $\mathbb{R}^2$. (Recall that the reflection over a line $L$ is given by $R_L = I_2 - 2P_L$.)
      [Hint: Write $S$ as $\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$, and use the projection formula to write $I_2$ and $P_L$ in this form as well. What is $L$?]

Solution.
   a) The only possible eigenvalues of an orthogonal matrix are $\pm 1$. If $S$ has only one eigenvalue then it is diagonal: it is diagonalizable, so that eigenvalue has geometric multiplicity 2.
   b) We can write $S = u_1 u_1^T - u_2 u_2^T$, where $u_1$ spans $L$ and $u_2$ spans $L^T$. On the other hand, $P_L = u_2 u_2^T$ and $I_2 = u_1 u_1^T + u_2 u_2^T$ by the projection formula, so $S = I_2 - 2P_L$.

7. a) Let $S$ be a diagonalizable (over $\mathbb{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that $S$ is symmetric.
   [Hint: choose orthonormal bases for each eigenspace.]
   b) Let $S$ be a matrix that can be written in the form
      \[ S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T \]
      for some vectors $q_1, q_2, \ldots, q_n$. Prove that $S$ is symmetric.
   c) Let $V$ be a subspace of $\mathbb{R}^n$, and let $P_V$ be the projection matrix onto $V$. Use a) or b) to prove that $P_V$ is symmetric. (We proved this in class using the formula $P_V = A(A^T A)^{-1} A^T$.)

Solution.
   a) By Gram–Schmidt, we can find an orthonormal basis of each eigenspace; collecting these bases together, by mutual orthogonality we get an orthonormal eigenbasis $\{q_1, q_2, \ldots, q_n\}$. Making a matrix $Q$ with these columns, we have $S = QDQ^T$ for a diagonal matrix $D$; taking transposes, we get $S^T = QD^T Q^T = QDQ^T = S$.
   b) Each matrix $q_i q_i^T$ is symmetric because $(q_i q_i^T)^T = q_i^T q_i = q_i q_i^T$, so $S$ is a sum of symmetric matrices.
   c) The 1-eigenspace of $P_V$ is $V$ and the zero-eigenspace is $V^\perp$. These are orthogonal, and the geometric multiplicities sum to $n$. 
8. For which matrices $A$ is $S = A^T A$ positive-definite? If $S$ is not positive-definite, find a vector $x$ such that $x^T S x = 0$. In any case, do not compute $S$!

\[ a) \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} \quad b) \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \quad c) \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \]

Solution.

a) This is positive-definite because $A$ has full column rank.

b) This is not positive-definite. The vector $x = (-6, 3, 1)$ is in the null space, so $x^T A^T A x = 0$.

c) This is not positive-definite. The vector $x = (1, -2, 1)$ is in the null space, so $x^T A^T A x = 0$.

9. a) If $S$ is positive-definite and $C$ is invertible, show that $C S C^T$ is positive-definite.

b) If $S$ and $T$ are positive-definite, show that $S + T$ is positive-definite.

c) If $S$ is positive-definite, show that $S$ is invertible and that $S^{-1}$ is positive-definite.

[Hint: For a) and b) use the positive-energy characterization of positive-definiteness; for c) use the positive-eigenvalue characterization.]

Solution.

a) Let $x$ be a nonzero vector. Then

\[ x^T C S C^T x = (C^T x)^T S (C^T x), \]

which is positive because $C^T x \neq 0$.

b) Let $x$ be a nonzero vector. Then

\[ x^T (S + T)x = x^T S x + x^T T x, \]

which is positive because $x^T S x > 0$ and $x^T T x > 0$.

c) All eigenvalues of $S$ are positive, as are their reciprocals, which are the eigenvalues of $S^{-1}$.

10. Let $S$ be a positive-definite matrix.

a) Show that the diagonal entries of $S$ are positive.

[Hint: compute $e_i^T S e_i$.]

b) Show that the diagonal entries of $S$ are all greater than or equal to the smallest eigenvalue of $S$.

[Hint: if not, apply a) to $S - a I_n$ for a diagonal entry $a$ that is smaller than all eigenvalues.]

Solution.
a) If $s_{ij}$ is the $(i, j)$-entry of $S$, then $s_{ii} = e_i^T S e_i > 0$.

b) Suppose that $S$ has a diagonal entry $a$ that is smaller than the smallest eigenvalue $\lambda$. The eigenvalues of $S - aI_n$ have the form $\lambda - a$, where $\lambda$ is an eigenvalue of $S$, hence are all positive numbers. Since $S - aI_n$ has positive eigenvalues, it is positive-definite, which is not possible by a) because it has a zero on the diagonal. It follows that no such diagonal entry exists.

11. Decide if each statement is true or false, and explain why. All matrices are real.

a) A symmetric matrix is diagonalizable.

b) If $A$ is any matrix then $A^T A$ is positive-semidefinite.

c) A symmetric matrix with positive determinant is positive-definite.

d) If $A = C D C^{-1}$ for a diagonal matrix $D$ and a non-orthogonal invertible matrix $C$, then $A$ is not symmetric.

e) A positive-definite matrix has the form $A^T A$ for a matrix $A$ with full column rank.

f) The only positive-definite projection matrix is the identity.

g) All eigenvalues of a positive-definite symmetric matrix are positive real numbers.

Solution.

a) True: this is part of the Spectral Theorem.

b) True.

c) False: you need all upper-left determinants to be positive.

d) False: you can always find an orthonormal eigenbasis, but you don't have to use it to diagonalize a symmetric matrix.

e) True.

f) True.

g) True.
12. For each symmetric matrix $S$, decide if $S$ is positive-definite. If so, find its $LDL^T$ and Cholesky decompositions. Do not compute any eigenvalues!

$$a) \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \quad b) \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \quad c) \begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$d) \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix} \quad e) \begin{pmatrix} -1 & 2 & 3 & -2 \\ -2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$$

**Solution.**

a) This is positive-definite, and $S = LDL^T = L_1L_1^T$ for

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad L_1 = \begin{pmatrix} 1 & 0 \\ 1 & \sqrt{2} \end{pmatrix}.$$

b) This is positive-definite, and $S = LDL^T = L_1L_1^T$ for

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & \sqrt{2} \end{pmatrix}.$$

c) This is positive-semidefinite but not positive-definite.

d) This is positive-definite, and $S = LDL^T = L_1L_1^T$ for

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}.$$

e) This is not positive-definite.

13. Consider the matrix

$$S = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$ 

*Without multiplying* the matrices, find:

a) The determinant of $S$.

b) The eigenvalues of $S$.

c) The eigenvectors of $S$.

d) A reason why $S$ is symmetric positive-definite.

**Solution.**

This matrix has the form $S = QDQ^T$ for an orthogonal matrix $Q$.

a) The determinant is $3 \cdot 4 = 12$. 
b) The eigenvalues are 3 and 4.
c) The eigenvectors are nonzero multiples of \((\cos \theta, \sin \theta)\) and \((-\sin \theta, \cos \theta)\).
d) The eigenvalues are positive.

14. a) For each symmetric matrix \(S\), compute the associated quadratic form \(q(x) = x^T S x\).

\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 & 3 \\
0 & -1 & 1 \\
3 & 1 & 0
\end{pmatrix}
\]

b) Let \(A\) be a square matrix and let \(S = \frac{1}{2}(A + A^T)\). Show that \(S\) is symmetric and that \(x^T A x = x^T S x\). (This is why we only consider symmetric matrices when studying quadratic forms.)

Solution.

a) They are:

\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix} : \quad q(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2\\
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} : \quad q(x_1, x_2) = 2x_1x_2\\
\begin{pmatrix}
1 & 0 & 3 \\
0 & -1 & 1 \\
3 & 1 & 0
\end{pmatrix} : \quad q(x_1, x_2, x_3) = x_1^2 - x_2^2 + 6x_1x_3 + 2x_2x_3
\]

b) We have \(S^T = \frac{1}{2}(A^T + A) = S\), so \(S\) is symmetric. The “matrix” \(x^T A^T x\) is just a number, so it is symmetric, and hence \(x^T A^T x = x^T S x\). Therefore,

\[
x^T \frac{1}{2}(A + A^T) x = \frac{1}{2} x^T A x + \frac{1}{2} x^T A^T x = x^T A x.
\]

15. For each quadratic form \(q(x_1, x_2)\), i) write \(q(x)\) in the form \(x^T S x\) for a symmetric matrix \(S\), ii) find coordinates \(y_1, y_2\) such that \(q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2\), and iii) find the maximum and minimum values of \(q(x_1, x_2)\) subject to the constraint \(x_1^2 + x_2^2 = 1\), and at which points \((x_1, x_2)\) these values are attained.

a) \(q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2\)

b) \(q(x_1, x_2) = \frac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2)\)

c) \(q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2\)

Solution.

a) We have \(q = x^T S x\) for \(S = \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}\). This is the matrix of Problem 1(c); we found its orthogonal diagonalization to be \(S = Q D Q^T\) for

\[
Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix}.
\]
Taking \( x = Qy \), we have \( q(x) = 15y_1^2 + 10y_2^2 \). The minimum value is \( q(\pm 1/\sqrt{5}, \pm 2/\sqrt{5}) = 10 \) and the maximum value is \( q(\pm 2/\sqrt{5}, \pm 1/\sqrt{5}) = 15 \).

b) We have \( q = x^T S x \) for \( S = \frac{1}{10} \begin{pmatrix} 21 & -3 \\ -3 & 29 \end{pmatrix} \). This has eigenvalues 3 and 2 with unit eigenvectors \( w_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix} \) and \( w_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \), respectively. Therefore, \( S = QDQ^T \) for

\[
Q = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.
\]

Taking \( x = Qy \), we have \( q(x) = 3y_1^2 + 2y_2^2 \). The minimum value is \( q(\pm 3/\sqrt{10}, \pm 1/\sqrt{10}) = 2 \) and the maximum value is \( q(\pm 1/\sqrt{10}, \pm 3/\sqrt{10}) = 3 \).

c) We have \( q = x^T S x \) for \( S = \begin{pmatrix} 1 & -3 \\ -3 & -1 \end{pmatrix} \). This is the matrix of Problem 1(a); we found its orthogonal diagonalization to be \( S = QDQ^T \) for

\[
Q = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad D = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}.
\]

Taking \( x = Qy \), we have \( q(x) = 4y_1^2 - 2y_2^2 \). The maximum value is \( q(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = 4 \), and the minimum value is \( q(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}) = -2 \).

16. For the quadratic form

\[ q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3, \]

find coordinates \( y_1, y_2, y_3 \) such that \( q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \), and find the maximum and minimum values of \( q(x_1, x_2, x_3) \) subject to the constraint \( x_1^2 + x_2^2 + x_3^2 = 1 \), along with the points \( (x_1, x_2, x_3) \) at which these values are attained.

**Solution.**

We have \( q(x) = x^T S x \) for

\[
S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}.
\]

This is the matrix of Problem 1(d); we found its orthogonal diagonalization to be \( S = QDQ^T \) for

\[
Q = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}
\]

(we have rearranged the eigenvalues to be in decreasing order). Taking \( x = Qy \), we have \( q(x) = 9y_1^2 + 6y_2^2 + 3y_3^2 \). The maximum value is \( q(\pm 2/3, \pm 2/3, \pm 1/3) = 9 \) and the minimum value is \( q(\pm 1/3, \mp 2/3, \pm 1/3) = 3 \).

17. Consider the quadratic form

\[ q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3. \]

Find all vectors \( x = (x_1, x_2, x_3) \) maximizing \( q(x) \) subject to \( ||x|| = 1 \). (There are infinitely many!)
Solution.

We have $q(x) = x^T S x$ for

$$S = \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}.$$

This is the matrix of Problem 1(e); we found its orthogonal diagonalization to be $S = Q D Q^T$ for

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{pmatrix} \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{pmatrix}.$$

The maximal value of $q$ is therefore equal to 9; it is attained at any unit 9-eigenvector. Letting $w_1 = \frac{1}{\sqrt{2}} (-1, 1, 0)$ and $w_2 = \frac{1}{3\sqrt{2}} (1, 1, 4)$, the maximum is achieved at any vector of the form

$$x = x_1 w_1 + x_2 w_2 \quad \text{for} \quad x_1^2 + x_2^2 = 1.$$