Homework #10
Answer Key

1. For each matrix $A$ and each vector $v$, decide if $v$ is an eigenvector of $A$, and if so, find the eigenvalue $\lambda$.

   a) \[
   \begin{pmatrix}
   -20 & 42 & 58 \\
   1 & -1 & -3 \\
   -1 & 18 & 26
   \end{pmatrix}, \quad \begin{pmatrix}
   1 \\
   5 \\
   2
   \end{pmatrix} \\
   \begin{pmatrix}
   1 & 5 \\
   2 & 3 \\
   4 & 6
   \end{pmatrix}, \quad \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]

   b) \[
   \begin{pmatrix}
   2 & 3 & 0 \\
   -5 & 4 & 2 \\
   3 & 3 & 3
   \end{pmatrix}, \quad \begin{pmatrix}
   1 \\
   3 \\
   3
   \end{pmatrix} \\
   \begin{pmatrix}
   2 & 3 & 0 \\
   3 & 4 & 2 \\
   3 & 3 & 3
   \end{pmatrix}, \quad \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]

   c) \[
   \begin{pmatrix}
   -7 & 32 \\
   7 & -22 \\
   3 & -11
   \end{pmatrix}, \quad \begin{pmatrix}
   3 \\
   -2 \\
   -1
   \end{pmatrix} \\
   \begin{pmatrix}
   -3 & 2 & -3 \\
   3 & -3 & -2 \\
   -4 & 2 & -3
   \end{pmatrix}, \quad \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]

   d) \[
   \begin{pmatrix}
   -20 & 42 & 58 \\
   1 & -1 & -3 \\
   -1 & 18 & 26
   \end{pmatrix}, \quad \begin{pmatrix}
   2 & 3 & 0 \\
   -5 & 4 & 2 \\
   3 & 3 & 3
   \end{pmatrix}, \quad \begin{pmatrix}
   0 \\
   0 \\
   0
   \end{pmatrix}
   \]

   Solution.
   a) not an eigenvector  b) yes, $\lambda = 5$  c) yes, $\lambda = -3$
   d) yes, $\lambda = 0$  e) not an eigenvector

2. Suppose that $A$ is an $n \times n$ matrix such that $Av = 2v$ for some $v \neq 0$. Let $C$ be any invertible matrix. Consider the matrices

   a) $A^{-1}$  b) $A + 2I_n$  c) $A^3$  d) $CAC^{-1}$.

   Show that $v$ is an eigenvector of a)–c) and that $Cv$ is as eigenvector of d), and find the eigenvalues.

   Solution.
   a) $Av = 2v \implies A^{-1}(2v) = v \implies A^{-1}v = \frac{1}{2}v$.
   b) $(A + 2I_n)v = Av + 2v = 2v + 2v = 4v$.
   c) $A^3v = A^2(Av) = 2A^2v = 2A(2v) = 4Av = 8v$.
   d) $(CAC^{-1})Cv = CAv = C(2v) = 2Cv$.

3. Here is a handy trick for computing eigenvectors of a $2 \times 2$ matrix.

   Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a $2 \times 2$ matrix with eigenvalue $\lambda$. Explain why $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ and $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$ are $\lambda$-eigenvectors of $A$ if they are nonzero.

   For which matrices $A$ does this trick fail?

   Solution.
   Since $\lambda$ is an eigenvalue of $A$, the matrix $A - \lambda I_2$ is not invertible, so its rows are multiples of each other. The first row has zero dot product with $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$, and the second has zero dot product with $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$.
This will fail when both \( \binom{-b}{a-\lambda} \) and \( \binom{d-\lambda}{c} \) are zero, which happens when \( A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I_2 \), in which case any nonzero vector is a \( \lambda \)-eigenvector.

4. For each \( 2 \times 2 \) matrix \( A \), i) compute the characteristic polynomial using the formula \( p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) \). Use this to ii) find all real eigenvalues, and iii) find a basis for each eigenspace, using Problem 3 when applicable. iv) Draw and label each eigenspace. v) Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

\[
\begin{align*}
\text{a)} & \quad \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \\
\text{b)} & \quad \begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix} \\
\text{c)} & \quad \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \\
\text{d)} & \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\
\text{e)} & \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]

Solution.

\text{a)} \quad p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3). The eigenvalues are 2 and 3, with eigenspaces \( \text{Span}\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\} \) and \( \text{Span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\} \), respectively.

This matrix is equal to \( CDC^{-1} \) for
\[
C = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.
\]

\text{b)} \quad p(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2. The only eigenvalue is 2, and the 2-eigenspace is \( \text{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \).

This matrix is not diagonalizable.

\text{c)} \quad p(\lambda) = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2. The only eigenvalue is 3, and the 3-eigenspace is \( \mathbb{R}^2 \).

This matrix is diagonal; it is equal to \( CDC^{-1} \) for \( D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \) and \( C = I_2 \).
d) \( p(\lambda) = \lambda^2 - 2\lambda + 2 \). This polynomial has no real roots, so there are no real eigenvalues.

\[ \lambda = \lambda^2 - 2\lambda + 2 = 0 \]

(no eigenvectors)

In particular, it is not diagonalizable (over the real numbers).

e) \( p(\lambda) = \lambda^2 - \lambda - 1 \). The eigenvalues are \( \frac{1}{2}(1 + \sqrt{5}) \) and \( \frac{1}{2}(1 - \sqrt{5}) \), with eigenspaces \( \text{Span}\left\{ \left( \begin{array}{c} -1 \\ 1-\sqrt{5}/2 \end{array} \right) \right\} \) and \( \text{Span}\left\{ \left( \begin{array}{c} -1 \\ 1+\sqrt{5}/2 \end{array} \right) \right\} \), respectively.

This matrix is equal to \( CDC^{-1} \) for

\[
C = \begin{pmatrix} -1 & -1 \\ 1-\sqrt{5}/2 & 1+\sqrt{5}/2 \end{pmatrix}, \quad D = \frac{1}{2} \begin{pmatrix} 1+\sqrt{5} & 0 \\ 0 & 1-\sqrt{5} \end{pmatrix}.
\]

5. For each matrix \( A \), i) find all real eigenvalues of \( A \), and ii) find a basis for each eigenspace. iii) Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix \( C \) and a diagonal matrix \( D \) such that \( A = CDC^{-1} \).

At least one eigenvalue is provided; use that and synthetic division (or a computer algebra system) to find the others.

a) \( \begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix} \), \( \lambda = -1 \)  \hspace{1cm} b) \( \begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix} \), \( \lambda = 1 \)

c) \( \begin{pmatrix} 6 & 2 & 3 \\ -14 & -7 & -12 \\ 1 & 2 & 4 \end{pmatrix} \), \( \lambda = 1 \)

Optional (if you want more practice):

d) \( \begin{pmatrix} -11 & -54 & 10 \\ -2 & -7 & 2 \\ -21 & -90 & 20 \end{pmatrix} \), \( \lambda = 1 \)  \hspace{1cm} e) \( \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \), \( \lambda = 2 \)

f) \( \begin{pmatrix} 13 & 18 & -18 \\ -12 & -17 & 18 \\ -4 & -6 & 7 \end{pmatrix} \), \( \lambda = 1 \)  \hspace{1cm} g) \( \begin{pmatrix} -10 & 28 & -18 & -76 \\ -1 & 9 & -6 & -2 \\ 4 & -8 & 7 & 26 \\ 0 & 2 & -2 & 4 \end{pmatrix} \), \( \lambda_1 = 1 \)  \hspace{1cm} \lambda_2 = 2
Solution.

a) \( p(\lambda) = -\lambda^3 + 4\lambda^2 - \lambda - 6 = -(\lambda - 3)(\lambda - 2)(\lambda + 1) \). The eigenvalues are 3, 2, and -1, with eigenspaces spanned by \((-1, -2, 2), (-3, -2, 1), \) and \((1, 0, 0), \) respectively. This matrix is equal to \( CDC^{-1} \) for

\[
C = \begin{pmatrix} -1 & -3 & 1 \\ -2 & -2 & 0 \\ 2 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

b) \( p(\lambda) = -\lambda^3 - \lambda^2 + \lambda + 1 = -(\lambda - 1)(\lambda + 1)^2 \). The eigenvalues are 1 and -1, with eigenspaces

\[
\text{Span}\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \text{Span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} \right\},
\]

respectively. This matrix is equal to \( CDC^{-1} \) for

\[
C = \begin{pmatrix} 2 & -3 & -3 \\ -2 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

c) \( p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 \). The only eigenvalue is 1, and the 1-eigenspace is

\[
\text{Span}\left\{ \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right\}.
\]

This matrix is not diagonalizable.

d) \( p(\lambda) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda - 1)(\lambda - 2)(\lambda + 1) \). The eigenvalues are 1, 2, and -1, with eigenspaces spanned by \((7, -1, 3), (-6, 2, 3), \) and \((1, 0, 1), \) respectively. This matrix is equal to \( CDC^{-1} \) for

\[
C = \begin{pmatrix} 7 & -6 & 1 \\ -1 & 2 & 0 \\ 3 & 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

e) \( p(\lambda) = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda - 2)^3 \). The only eigenvalue is 2, and the 2-eigenspace is \( \mathbb{R}^3 \). The matrix is diagonal; it is equal to \( CDC^{-1} \) for \( D = 2I_3 \) and \( C = I_3 \).

f) \( p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 \). The only eigenvalue is 1, and the 1-eigenspace is spanned by \((0, -3, 2)\). This matrix is not diagonalizable.

g) \( p(\lambda) = \lambda^4 - 10\lambda^3 + 35\lambda^2 - 50\lambda + 24 = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) \). The eigenvalues are 1, 2, 3, and 4, with eigenspaces spanned by \((-16, 3, 6, 2), (-7, 1, 2, 1), (-10, 4, 5, 2), \) and \((-4, 2, 2, 1), \) respectively. This matrix is equal to \( CDC^{-1} \) for

\[
C = \begin{pmatrix} -16 & -7 & -10 & -4 \\ 3 & 1 & 4 & 2 \\ 6 & 2 & 5 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}.
\]
6. Consider the matrix
\[ A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}. \]

a) Find a diagonal matrix \( D \) and an invertible matrix \( C \) such that \( A = CDC^{-1} \).

b) Find a different diagonal matrix \( D' \) and a different invertible matrix \( C' \) such that \( A = C'D'C'^{-1} \).

[Hint: Try re-ordering the eigenvalues.]

Solution.

a) \( C = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \)

b) \( C' = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad D' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

7. Compute the matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors
\( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \).

(There is only one such matrix.)

Solution.
\[ \begin{pmatrix} -2 & 2 & -1 \\ -2 & 2 & 0 \\ 2 & -2 & 3 \end{pmatrix} \]

8. a) Show that \( A \) and \( A^T \) have the same eigenvalues.

b) Give an example of a \( 2 \times 2 \) matrix \( A \) such that \( A \) and \( A^T \) do not share any eigenvectors.

c) A stochastic matrix is a matrix with nonnegative entries whose columns sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix.

[Hint: show that \( (1, 1, \ldots, 1) \) is an eigenvector of \( A^T \).]

Solution.

a) They have the same characteristic polynomial:
\[ \det(A^T - \lambda I_n) = \det((A - \lambda I_n)^T) = \det(A - \lambda I_n). \]

b) There are many answers. For instance, the matrix \( A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} \) of Problem 4(a) has eigenspaces spanned by \( \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), but \( A^T = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \) has eigenspaces spanned by \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \).
c) The condition on the columns means
\[ A^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \]
Hence \((1, 1, \ldots, 1)\) is an eigenvector of \(A^T\) with eigenvalue 1.

9. a) Find all eigenvalues of the matrix
\[
\begin{pmatrix}
1 & -1 & 2 & 3 & 4 \\
0 & 3 & -1 & -2 & -5 \\
0 & 0 & 1 & 2 & 4 \\
0 & 0 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

b) Explain how to find the eigenvalues of any triangular matrix.

Solution.
a) The matrix \(A - \lambda I_5\) is upper-triangular, so its determinant is the product of its diagonal entries:
\[
p(\lambda) = \det(A - \lambda I_5) = (1 - \lambda)(3 - \lambda)(3 - \lambda)(2 - \lambda)(-1 - \lambda).
\]
Hence the eigenvalues are just the diagonal entries: \(1, 3, 1, 2, -1\).

b) The above argument works for any triangular matrix.

10. Let \(A\) be an \(n \times n\) matrix, and let \(C\) be an invertible \(n \times n\) matrix. Prove that the characteristic polynomial of \(CAC^{-1}\) equals the characteristic polynomial of \(A\).

In particular, \(A\) and \(CAC^{-1}\) have the same eigenvalues, the same determinant, and the same trace. They are called similar matrices.

Solution.
First we note that \(C(A - \lambda I_n)C^{-1} = (CA - \lambda C)C^{-1} = CAC^{-1} - \lambda I_n\), so
\[
\det(CAC^{-1} - \lambda I_n) = \det(C(A - \lambda I_n)C^{-1}) = \det(C) \det(A - \lambda I_n) \det(C)^{-1} = \det(A - \lambda I_n).
\]

11. Recall that an orthogonal matrix is a square matrix with orthonormal columns. Prove that any real eigenvalue of an orthogonal matrix \(Q\) is \(\pm 1\).

Solution.
We have \(\|Qx\| = \|x\|\) for all \(x\), so if \(Qx = \lambda x\) then \(|\lambda| = 1\).

12. Let \(V\) be the plane \(x + y + z = 0\), and let \(R_V = I_3 - 2P_{V^\perp}\) be the reflection matrix over \(V\), as in HW9#11. Diagonalize \(R_V\) without doing any computations.

Solution.
In HW9#11 we computed

\[ R_V = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}. \]

Any vector on the plane \( V \) is fixed by \( R_V \), so \((-1, 1, 0)\) and \((-1, 0, 1)\) are 1-eigenvectors. Any vector on \( V^\perp \) is negated by \( R_V \), so \((1, 1, 1)\) is a \(-1\)-eigenvector. Hence \( R_V = CDC^{-1} \) for

\[ C = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

13. The Fibonacci numbers are defined recursively as follows:

\[ F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0). \]

The first few Fibonacci numbers are \(0, 1, 1, 2, 3, 5, 8, 13, \ldots\) In this problem, you will find a closed formula (as opposed to a recursive formula) for the \( n \)th Fibonacci number by solving a difference equation.

\( \text{a)} \) Let \( v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \), so \( v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), etc. Find a state change matrix \( A \) such that \( v_{n+1} = Av_n \) for all \( n \geq 0 \).

\( \text{b)} \) Show that the eigenvalues of \( A \) are \( \lambda_1 = \frac{1}{2}(1 + \sqrt{5}) \) and \( \lambda_2 = \frac{1}{2}(1 - \sqrt{5}) \), with corresponding eigenvectors \( w_1 = \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix} \) and \( w_2 = \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix} \).

[Hint: Check that \( Aw_i = \lambda_i w_i \) using the relations \( \lambda_1 \lambda_2 = -1 \) and \( \lambda_1 + \lambda_2 = 1 \).]

\( \text{c)} \) Expand \( v_0 \) in this eigenbasis: that is, find \( x_1, x_2 \) such that \( v_0 = x_1 w_1 + x_2 w_2 \). (It helps to write \( x_1, x_2 \) in terms of \( \lambda_1, \lambda_2 \).)

\( \text{d)} \) Multiply \( v_0 = x_1 w_1 + x_2 w_2 \) by \( A^n \) to show that

\[ F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \]

\( \text{e)} \) Use this formula to explain why \( F_{n+1}/F_n \) approaches the golden ratio when \( n \) is large.

**Solution.**

\( \text{a)} \) We have the system of equations

\[ F_{n+2} = F_{n+1} + F_n \]
\[ F_{n+1} = F_{n+1}, \]

which translates into the matrix equation

\[ v_{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} v_n. \]
b) We did this in Problem 4(e). We can also verify the eigenvectors directly using the relations $\lambda_1 \lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$:

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 + \lambda_2 \\ -1 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix} \\
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} -1 + \lambda_1 \\ -1 \end{pmatrix} = \lambda_2 \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix}.
\]

c) We can solve this by inspection:

\[
\begin{pmatrix} \lambda_2 - \lambda_1 \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix} - \lambda_2 \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix} \implies v_0 = \frac{\lambda_1 w_1 - \lambda_2 w_2}{\lambda_2 - \lambda_1}.
\]

d) Multiplying the above equation by $A^n$ gives

\[
\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = v_n = \frac{\lambda_1^{n+1} w_1 - \lambda_2^{n+1} w_2}{\lambda_2 - \lambda_1}.
\]

The second coordinate of this is $F_n$; this works out to be

\[
F_n = \frac{\lambda_2 \lambda_1^{n+1} - \lambda_1 \lambda_2^{n+1}}{\lambda_2 - \lambda_1} = \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1}.
\]

e) Since $|\lambda_1| > |\lambda_2|$, when $n$ is large we have $\lambda_1^n - \lambda_2^n \approx \lambda_1^n$, so

\[
\frac{F_{n+1}}{F_n} = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1^n - \lambda_2^n} \approx \frac{\lambda_1^{n+1}}{\lambda_1^n} = \lambda_1.
\]

14. Pretend that there are three Red Box kiosks in Durham. Let $x_t, y_t, z_t$ be the number of copies of Prognosis Negative at each of the three kiosks, respectively, on day $t$. Suppose in addition that a customer renting a movie from kiosk $i$ will return the movie the next day to kiosk $j$, with the following probabilities:

<table>
<thead>
<tr>
<th>Returning to kiosk</th>
<th>Renting from kiosk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 2 3</td>
</tr>
<tr>
<td>1</td>
<td>30% 40% 50%</td>
</tr>
<tr>
<td>2</td>
<td>30% 40% 30%</td>
</tr>
<tr>
<td>3</td>
<td>40% 20% 20%</td>
</tr>
</tbody>
</table>

For instance, a customer renting from kiosk 3 has a 50% probability of returning it to kiosk 1.

a) Let $v_t = (x_t, y_t, z_t)$. Find the state change matrix $A$ such that $v_{t+1} = Av_t$.

b) Diagonalize $A$. What are its eigenvalues?

[Hint: $A$ is a stochastic matrix, so you know one eigenvalue by Problem 8(c).]

c) If you start with a total of 1 000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk? Does it matter what the initial state is?
This is an example of a stochastic process, and is an important application of eigenvalues and eigenvectors.

Solution.

\[
A = \begin{pmatrix}
0.3 & 0.4 & 0.5 \\
0.3 & 0.4 & 0.3 \\
0.4 & 0.2 & 0.2 \\
\end{pmatrix}
\]

a) The eigenvalues of \( A \) are 1, -.2, and .1, with eigenvectors

\[
w_1 = \begin{pmatrix} 1.4 \\ 1.2 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0.5 \\ -1.5 \\ 1 \end{pmatrix}.
\]

b) If the starting state is \( v_0 = x_1w_1 + x_2w_2 + x_3w_3 \), then

\[
A^n v_0 = x_1w_1 + (-.2)^n x_2w_2 + (.1)^n x_3w_3.
\]

Since \((- .2)^n \to 0 \) and \((.1)^n \to 0 \) as \( n \to \infty \), this approaches \( x_1w_1 \) as \( n \to \infty \). Hence the movies will be distributed in a 1.4 : 1.2 : 1 ratio; in percentages, this is approximately 38.9%, 33.3%, and 27.8%. Therefore, there will eventually be \( \approx 389 \) movies in kiosk 1, \( \approx 333 \) movies in kiosk 2, and \( \approx 278 \) in kiosk 3. (This does not depend on the initial state \( v_0 \).)

15. Let \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \). Find a closed formula for \( A^n \): that is, an expression of the form

\[
A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},
\]

where \( a_{ij}(n) \) is a function of \( n \).

Solution.

This is a diagonalization problem: \( A = CDC^{-1} \) for

\[
C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix},
\]

so

\[
A^n = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 + 3^n & 1 - 3^n \\ 1 - 3^n & 1 + 3^n \end{pmatrix}.
\]

16. Give an example of each of the following, or explain why no such example exists.

a) An invertible matrix with characteristic polynomial \( p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda \).

b) A 2 \( \times \) 2 orthogonal matrix with no real eigenvalues.

Solution.

a) Does not exist: \( \det(A) = p(0) = 0 \).

b) \( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \).
17. Suppose that $A$ is a square matrix such that $A^k$ is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of $A$.

**Solution.**
If $Av = \lambda v$ then $0 = A^k v = \lambda^k v$, so $\lambda^k = 0$ and hence $\lambda = 0$.

18. Decide if each statement is true or false, and explain why.

a) If $v, w$ are eigenvectors of a matrix $A$, then so is $v + w$.

b) An eigenvalue of $A + B$ is the sum of an eigenvalue of $A$ and an eigenvalue of $B$.

c) An eigenvalue of $AB$ is the product of an eigenvalue of $A$ and an eigenvalue of $B$.

d) If $Ax = \lambda x$ for some vector $x$, then $\lambda$ is an eigenvalue of $A$.

e) A matrix with eigenvalue 0 is not invertible.

f) The eigenvalues of $A$ are equal to the eigenvalues of a row echelon form of $A$.

g) If $v, w$ are eigenvectors of $A$ with different eigenvalues, then $\{v, w\}$ is linearly independent.

**Solution.**
a) False: they need to have the same eigenvalue for this to be true.
b) False.
c) False.
d) False: the vector needs to be nonzero.
e) True.
f) False.
g) True.