MATH 218D-1
MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

Problem 1. [20 points]

Consider the difference equation

$$\begin{aligned} x_{k+1} &= 3x_k + 8z_k \\ y_{k+1} &= -5x_k + 2y_k + 3z_k \\ z_{k+1} &= -x_k + y_k. \end{aligned}$$

a) Setting $v_k = (x_k, y_k, z_k)$, find a matrix A such that $v_{k+1} = Av_k$.

$$A = \begin{pmatrix} 3 & 0 & 8 \\ -5 & 2 & 3 \\ -1 & 1 & 0 \end{pmatrix}$$

b) Compute the characteristic polynomial of the matrix you found in **a)**. *Do not factor* $p(\lambda)$.

$$p(\lambda) = \boxed{-\lambda^3 + 5\lambda^2 - 11\lambda - 33}$$

Problem 1, continued.

Now we change matrices to avoid carry-through error. Consider the matrix

$$B = \begin{pmatrix} 3 & 8 & -28 \\ -5 & 0 & 13 \\ -1 & 1 & 0 \end{pmatrix}.$$

c) Find an invertible matrix C and a diagonal matrix D such that $B = CDC^{-1}$. The eigenvalues of B are 3, 1, and -1.

[**Hint:** Choose the ± 1 entry as your pivot when doing elimination!]

$$C = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

d) Solve the difference equation

$$v_{k+1} = Bv_k \qquad v_0 = \begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}.$$

$$v_k = \begin{pmatrix} 2 + (-1)^k 3 \\ 3 + (-1)^k 2 \\ 1 + (-1)^k \end{pmatrix}$$

Problem 2.

[15 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

Its eigenvalues are 7 and −2.

a) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.
[**Hint:** It helps to scale your eigenvectors to have integer entries before normalizing.]

$$Q = \begin{pmatrix} -1/\sqrt{5} & 4/3\sqrt{5} & -2/3 \\ 2/\sqrt{5} & 2/3\sqrt{5} & -1/3 \\ 0 & 5/3\sqrt{5} & 2/3 \end{pmatrix}$$

$$D = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

b) What are the algebraic and geometric multiplicities of each eigenvalue?

$$\lambda = 7: \quad \text{AM} = \boxed{2} \qquad \text{GM} = \boxed{2}$$

$$\lambda = -2: \quad \text{AM} = \boxed{1} \qquad \text{GM} = \boxed{1}$$

c) What is the *minimum* value of $q(x) = x^T Sx$ subject to $\|x\| = 1$, and at which vectors is the minimum achieved?

$$\text{Minimum} = \boxed{-2} \qquad \text{at} \quad x = \pm\frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$$

d) Find a matrix A such that $S = A^T A$, or explain why no such matrix exists.

No such matrix exists because S is indefinite.

Problem 3. [15 points]

Consider the initial value problem

$$u' = Au, \qquad u(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \text{for} \quad A = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}.$$

a) The matrix A has complex eigenvalues $\lambda = \boxed{\frac{1}{2}(1+i\sqrt{3})}$ and $\overline{\lambda} = \boxed{\frac{1}{2}(1-i\sqrt{3})}$.

b) Find eigenvectors w and $\overline{w}$ for λ and $\overline{\lambda}$, respectively.

$$w = \begin{pmatrix} 3 \\ \frac{1}{2}(3-i\sqrt{3}) \end{pmatrix} \qquad \overline{w} = \begin{pmatrix} 3 \\ \frac{1}{2}(3+i\sqrt{3}) \end{pmatrix}$$

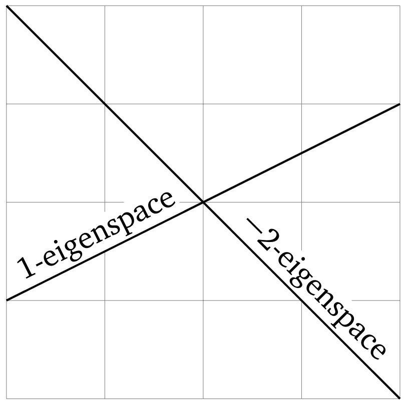
c) Solve the initial value problem. Your answer should only contain real numbers.

$$u = e^{t/2} \begin{pmatrix} 3\cos\frac{\sqrt{3}}{2}t - \sqrt{3}\sin\frac{\sqrt{3}}{2}t \\ 2\cos\frac{\sqrt{3}}{2}t \end{pmatrix}$$

Problem 4.

[10 points]

Find the 2×2 matrix A whose eigenspaces are drawn below. The grid is square.



$$A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Problem 5. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. (No justification is needed if an example does exist.)

All matrices in this problem have real entries.

a) A matrix with characteristic polynomial $p(\lambda) = -(\lambda-2)(\lambda-3)^2$ whose 2-eigenspace is a plane.

No such matrix exists: the geometric multiplicity cannot be greater than the algebraic multiplicity.

b) A 2×2 diagonalizable matrix with only one eigenvalue.

The only answers are multiples of the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

c) A 3×3 symmetric matrix that is diagonalizable over the complex numbers but not over the real numbers.

No such matrix exists by the spectral theorem.

d) A 3×3 matrix with no real eigenvalues.

No such matrix exists: any cubic polynomial has a real root.

e) A diagonalizable 2×2 matrix with characteristic polynomial $p(\lambda) = \lambda^2$.

The only example is the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Problem 6.

[20 points]

True/false problems: **circle** the correct answer. No justification is needed. *All matrices in this problem have real entries.*

a) (T) F — A matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 3\lambda^2 - 2\lambda - 2$ is invertible.

b) (T) F — If A is a square matrix and x is a nonzero vector in Nul(A), then x is an eigenvector of A.

c) T (F) — The eigenvalues of a square matrix are the diagonal entries.

d) T (F) — If S is symmetric, then either S or $-S$ is positive-semidefinite.

e) T (F) — Every 2×2 matrix is diagonalizable over the complex numbers.