Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the quadratic form

\[ q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 8x_1x_3 + 8x_2x_3. \]

a) Find a symmetric matrix \( S \) such that \( q(x) = x^T S x \).

b) Compute the characteristic polynomial \( p(\lambda) \) of \( S \). The roots of \( p(\lambda) \) are 9, 3, and -3.

c) Find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( S = QDQ^T \).

d) Find a change of coordinates \( y_1, y_2, y_3 \) such that

\[ q(x_1, x_2, x_3) = 9y_1^2 + 3y_2^2 - 3y_3^2. \]

(The \( x_i \) should be linear functions of the \( y_i \)).

e) What are the minimum and maximum values of \( q(x) \) subject to \( \|x\| = 1 \)? For which values of \( x \) are those values attained?

Your answers should involve square roots and fractions, not decimals.
Problem 2. [10 points]

Consider the symmetric matrix

\[ S = \begin{pmatrix}
  1 & 2 & 0 \\
  2 & 6 & -2 \\
  0 & -2 & 5 \\
\end{pmatrix}. \]

a) Verify that $S$ is positive-definite without finding its eigenvalues.

b) Compute the $LDL^T$ and Cholesky decompositions of $S$:

\[ S = LDL^T \quad S = L_1L_1^T. \]
Problem 3. [20 points]

Consider the difference equation
\[
\begin{align*}
    x_{n+1} &= 2x_n - y_n & x_0 &= 1 \\
    y_{n+1} &= \frac{3}{2}x_n - \frac{1}{2}y_n & y_0 &= 2.
\end{align*}
\]

a) Find a matrix \( A \) such that
\[
A \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.
\]

b) Find the eigenvalues of \( A \), and find corresponding eigenvectors.

c) Find a formula for \( \begin{pmatrix} x_n \\ y_n \end{pmatrix} \) in terms of \( n \).

d) What is \( \lim_{n \to \infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \)?

e) Solve the following initial value problem:
\[
\begin{align*}
    u_1'(t) &= 2u_1(t) - u_2(t) & u_1(0) &= 1 \\
    u_2'(t) &= \frac{3}{2}u_1(t) - \frac{1}{2}u_2(t) & u_2(0) &= 2.
\end{align*}
\]
Problem 4. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. All matrices in this problem have real entries.

a) A symmetric matrix satisfying

\[
\begin{pmatrix}
1 & 2 \\
2 & 4 \\
3 & 6
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
= \begin{pmatrix}
2 \\
4 \\
6
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
= \begin{pmatrix}
-2 \\
-1
\end{pmatrix}.
\]

b) A 2 × 2 matrix whose 1-eigenspace is the line \(x + 2y = 0\) and whose 2-eigenspace is the line \(x + 3y = 0\).

c) A 2 × 2 matrix that is neither invertible nor diagonalizable.

d) A 2 × 2 non-invertible matrix with eigenvalue \(2 + 3i\).

e) A 2 × 2 matrix \(A\) that is diagonalizable over \(\mathbb{R}\), such that \(A^2\) is not diagonalizable.
Problem 5. [10 points]

A certain diagonalizable $2 \times 2$ matrix $A$ has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw $Ax$ and $Ay$ on the diagram.

b) For which vectors $u$ is $\|A^n u\|$ bounded? In other words, for which vectors $u$ does $\|A^n u\|$ not approach $\infty$ as $n \to \infty$?
Problem 6.

In this problem, you need not explain your answers; just write them in the spaces provided.

Let $A$ be an $n \times n$ matrix with real entries.

a) Which one of the following statements is correct?

1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.
2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.
3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.
4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

b) Which one of the following statements is not correct?

1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.
4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

c) Which of the following $3 \times 3$ matrices are necessarily diagonalizable over the real numbers? (List all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A symmetric matrix with two real eigenvalues.
3. A matrix with a real eigenvalue $\lambda$ of algebraic multiplicity 2, such that the $\lambda$-eigenspace has dimension 2.
4. A matrix with a real eigenvalue $\lambda$ such that the $\lambda$-eigenspace has dimension 2.

d) Suppose that the characteristic polynomial of $A$ is

\[ p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2. \]

Which of the following can you determine from this information? (Circle all that apply.)

(1) The number $n$.
(2) The trace of $A$.
(3) The determinant of $A$.
(4) The rank of $A$.
(5) Whether $A$ is symmetric.
(6) Whether $A$ is diagonalizable.
(7) The eigenvalues of $A$. 

[20 points]