Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the quadratic form
\[ q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 8x_1x_3 + 8x_2x_3. \]

a) Find a symmetric matrix \( S \) such that \( q(x) = x^TSx \).

b) Compute the characteristic polynomial \( p(\lambda) \) of \( S \).

The roots of \( p(\lambda) \) are 9, 3, and \(-3\).

c) Find an orthogonal matrix \( Q \) and a diagonal matrix \( D \) such that \( S = QDQ^T \).

d) Find a change of coordinates \( y_1, y_2, y_3 \) such that
\[ q(x_1, x_2, x_3) = 9y_1^2 + 3y_2^2 - 3y_3^2. \]
(The \( x_i \) should be linear functions of the \( y_i \).)

e) What are the minimum and maximum values of \( q(x) \) subject to \( \|x\| = 1 \)? For which values of \( x \) are those values attained?

Your answers should involve square roots and fractions, not decimals.

Solution.

a) \[ S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix} \]

b) \[ p(\lambda) = -\lambda^3 + 9\lambda^2 + 9\lambda - 81 \]

c) \[ Q = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \quad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \]

d) \[ x_1 = -\frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \quad x_2 = \frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 - \frac{1}{\sqrt{3}}y_3 \quad x_3 = \frac{2}{\sqrt{6}}y_1 + \frac{1}{\sqrt{3}}y_3 \]

e) The minimum value is \(-3\), which is attained at \( \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \). The maximum value is 9, which is attained at \( \pm \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \).
Problem 2. \[10\text{ points}\]

Consider the symmetric matrix

\[ S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}. \]

a) Verify that \( S \) is positive-definite without finding its eigenvalues.

b) Compute the \( LDL^T \) and Cholesky decompositions of \( S \):

\[ S = LDL^T \quad S = L_1L_1^T. \]

Solution.

a) This can be accomplished by finding the \( LU \) decomposition, which we do in b).

b) We have \( S = LDL^T \) for

\[ L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \]

We also have \( S = L_1L_1^T \) for

\[ L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{3} \end{pmatrix}. \]
Problem 3.  

Consider the difference equation
\[
\begin{align*}
x_{n+1} &= 2x_n - y_n \quad x_0 = 1 \\
y_{n+1} &= \frac{3}{2}x_n - \frac{1}{2}y_n \quad y_0 = 2.
\end{align*}
\]

a) Find a matrix \(A\) such that
\[
A \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.
\]

b) Find the eigenvalues of \(A\), and find corresponding eigenvectors.

c) Find a formula for \(\begin{pmatrix} x_n \\ y_n \end{pmatrix}\) in terms of \(n\).

d) What is \(\lim_{n \to \infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix}\)?

e) Solve the following initial value problem:
\[
\begin{align*}
u_1'(t) &= 2u_1(t) - u_2(t) \quad u_1(0) = 1 \\
u_2'(t) &= \frac{3}{2}u_1(t) - \frac{1}{2}u_2(t) \quad u_2(0) = 2.
\end{align*}
\]

Solution.

a) The matrix is \(A = \begin{pmatrix} 2 & -1 \\ 3/2 & -1/2 \end{pmatrix}\).

b) The eigenvalues are \(\lambda_1 = 1\) and \(\lambda_2 = 1/2\), and corresponding eigenvectors are \(w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(w_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}\).

c) We have \(\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -w_1 + w_2\), so
\[
\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -A^n w_1 + A^n w_2 = -w_1 + \frac{1}{2^n} w_2 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2^n} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.
\]

d) The limit is \(-\begin{pmatrix} 1 \\ 1 \end{pmatrix}\).

e) The solution is
\[
\begin{align*}
u_1(t) &= -e^t + 2e^{t/2} \\
u_2(t) &= -e^t + 3e^{t/2}.
\end{align*}
\]
Problem 4. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. All matrices in this problem have real entries.

a) A symmetric matrix satisfying

\[
S \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}.
\]

b) A $2 \times 2$ matrix whose 1-eigenspace is the line $x + 2y = 0$ and whose 2-eigenspace is the line $x + 3y = 0$.

c) A $2 \times 2$ matrix that is neither invertible nor diagonalizable.

d) A $2 \times 2$ non-invertible matrix with eigenvalue $2 + 3i$.

e) A $2 \times 2$ matrix $A$ that is diagonalizable over $\mathbb{R}$, such that $A^2$ is not diagonalizable.

Solution.

a) Does not exist: eigenvectors with different eigenvalues would have to be orthogonal.

b) This matrix satisfies

\[
A = \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}.
\]

c) One example is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

d) Does not exist: the other eigenvalue would be $2 - 3i$, so 0 is not an eigenvalue.

e) Does not exist: if $A = CD\cdot C^{-1}$ then $A^2 = CD^2C^{-1}$.
Problem 5. [10 points]

A certain diagonalizable $2 \times 2$ matrix $A$ has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw $Ax$ and $Ay$ on the diagram.

b) For which vectors $u$ is $\|A^n u\|$ bounded? In other words, for which vectors $u$ does $\|A^n u\|$ not approach $\infty$ as $n \to \infty$?

Solution.

b) Such a $u$ must be a 1-eigenvector.
Problem 6.

In this problem, you need not explain your answers; just write them in the spaces provided.

Let \( A \) be an \( n \times n \) matrix with real entries.

a) Which **one** of the following statements is correct?

1. An eigenvector of \( A \) is a vector \( v \) such that \( Av = \lambda v \) for a nonzero scalar \( \lambda \).
2. An eigenvector of \( A \) is a nonzero vector \( v \) such that \( Av = \lambda v \) for a scalar \( \lambda \).
3. An eigenvector of \( A \) is a nonzero scalar \( \lambda \) such that \( Av = \lambda v \) for some vector \( v \).
4. An eigenvector of \( A \) is a nonzero vector \( v \) such that \( Av = \lambda v \) for a nonzero scalar \( \lambda \).

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of \( A \) is a scalar \( \lambda \) such that \( A - \lambda I \) is not invertible.
2. An eigenvalue of \( A \) is a scalar \( \lambda \) such that \( (A - \lambda I)v = 0 \) has a solution.
3. An eigenvalue of \( A \) is a scalar \( \lambda \) such that \( Av = \lambda v \) for a nonzero vector \( v \).
4. An eigenvalue of \( A \) is a scalar \( \lambda \) such that \( \det(A - \lambda I) = 0 \).

c) Which of the following \( 3 \times 3 \) matrices are necessarily diagonalizable over the real numbers? (List all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A symmetric matrix with two real eigenvalues.
3. A matrix with a real eigenvalue \( \lambda \) of algebraic multiplicity 2, such that the \( \lambda \)-eigenspace has dimension 2.
4. A matrix with a real eigenvalue \( \lambda \) such that the \( \lambda \)-eigenspace has dimension 2.

d) Suppose that the characteristic polynomial of \( A \) is
\[
p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.\
\]
Which of the following can you determine from this information? (Circle all that apply.)

(1) The number \( n \).
(2) The trace of \( A \).
(3) The determinant of \( A \).
(4) The rank of \( A \).
(5) Whether \( A \) is symmetric.
(6) Whether \( A \) is diagonalizable.
(7) The eigenvalues of \( A \).