Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the plane

\[ V = \{(x, y, z) : x - y + 2z = 0\}. \]

a) Find a basis for \( V \).

\[
\begin{bmatrix}
\end{bmatrix}
\]

b) Find an orthogonal basis for \( V \).

\[
\begin{bmatrix}
\end{bmatrix}
\]

c) Use the projection formula and your answer to part b) to compute the orthogonal projection \( b_V \) of the vector \( b = (1, 1, -3) \) onto \( V \).

\[
b_V = \begin{bmatrix}
\end{bmatrix}
\]

d) Find a basis for \( V^\perp \).

\[
\begin{bmatrix}
\end{bmatrix}
\]

e) Find an orthogonal basis of \( \mathbb{R}^3 \) containing the basis vectors you found in b).

\[
\begin{bmatrix}
\end{bmatrix}
\]
Problem 2. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & 5 \\
-1 & 1 & -4 \\
-1 & 4 & -3 \\
1 & -4 & 7 \\
1 & 2 & 1
\end{pmatrix}.
\]

a) Find the QR decomposition of \( A \). You should get

\[
R = \begin{pmatrix}
\sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\
0 & 6 & -2 \\
0 & 0 & 4
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
\end{pmatrix}.
\]
b) Solve $R\tilde{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$ to find the least-squares solution of $Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$.

\[ \tilde{x} = \begin{pmatrix} \end{pmatrix} \]

c) Compute the matrix $P_V$ for projection onto $V = \text{Col}(A)$.

\[ P_V = \begin{pmatrix} \end{pmatrix} \]
Problem 3. [15 points]

Consider the data points

\[ b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix} \]

drawn below.

\[ \begin{array}{c}
\vdots \\
\vdots \\
\end{array} \]

a) Find the matrix \( A \) such that the least-squares solution \( \bar{x} = (C, D) \) of

\[ A \begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \]

gives the coefficients of the best-fit line \( y = Cx + D \).

\[ A = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \]

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

\[ y = \square x + \square \]
c) Compute the minimized vector $b_{V^\perp}$. What does $b_{V^\perp}$ represent in the original best-fit problem? (Here $V = \text{Col}(A)$.)

$$ b_{V^\perp} = \begin{pmatrix} \hline \end{pmatrix} $$

\[ \text{d) What is the best-fit line among all lines passing through the origin?} \]

$$ y = \square x $$
Problem 4. [12 points]

A line $V$ and a vector $b$ are drawn below. Draw and label:

a) The orthogonal projection $b_V$.

b) The projection onto the orthogonal complement $b_{V^\perp}$.

c) The vector $b - 2b_{V^\perp}$. 
Problem 5. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) \( \text{Nul} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix} \)

b) \( \text{Col} \begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix} \)

c) The subspace of all vectors in \( \mathbb{R}^4 \) whose entries sum to zero.

d) The line \( \{(t, 2t, 3t) : t \in \mathbb{R}\} \).

e) \( \mathbb{R}^3 \)
Problem 6. [16 points]

a) Let $A$ be an $m \times n$ matrix of rank $r$. Which of the following statements are equivalent to “$A$ has full row rank”?

1. $\text{Nul}(A^T) = \{0\}$
2. $n = r$
3. $\text{Col}(A) = \mathbb{R}^m$
4. $A$ has linearly independent columns
5. $A$ has a pivot in every row
6. $A$ is invertible
7. $Ax = b$ is consistent for every vector $b$

b) Explain why the projection matrix $P_V$ onto a subspace $V$ can be written as $QQ^T$ for some matrix $Q$ with orthonormal columns. (What is $Q$ in terms of $V$?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly dependent, but $v_3$ is not in $\text{Span}\{v_1, v_2\}$. Be sure to label which is $v_3$.

d) Give an example of a $4 \times 4$ matrix $A$ such that $\text{Nul}(A) = \text{Row}(A)$, or explain why no such matrix exists.