Please read all instructions carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the printed pages (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a calculator for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must show your work so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is not meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.
Problem 1. [20 points]

Consider the plane

\[ V = \{(x, y, z) : x - y + 2z = 0\}. \]

a) Find a basis for \( V \).

b) Find an orthogonal basis for \( V \).

c) Use the projection formula and your answer to part b) to compute the orthogonal projection \( b_V \) of the vector \( b = (1, 1, -3) \) onto \( V \).

d) Find a basis for \( V^\perp \).

e) Find an orthogonal basis of \( \mathbb{R}^3 \) containing the basis vectors you found in b).

Solution.

a) There are many answers. If you find the solutions of \( x - y + 2z = 0 \) in parametric vector form, you get

\[ \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}. \]

b) Running Gram–Schmidt on the above vectors gives

\[ \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}. \]

c) \[ b_V = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}. \]

d) Since \( V = \text{Nul} \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \), the orthogonal complement \( V^\perp \) is the row space of \( \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \):

\[ \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}. \]

e) We just add the vectors in b) and d):

\[ \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}. \]

(You could also notice that \( b - b_V = (-1, 1, -2) \) spans \( V^\perp \).)
Problem 2. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & 5 \\
-1 & 1 & -4 \\
-1 & 4 & -3 \\
1 & -4 & 7 \\
1 & 2 & 1
\end{pmatrix}.
\]

a) Find the QR decomposition of \(A\). You should get \(R = \begin{pmatrix}
\sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\
0 & 6 & -2 \\
0 & 0 & 4
\end{pmatrix}\).

b) Solve \(R\hat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}\) to find the least-squares solution of \(Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}\).

c) Compute the matrix \(P_v\) for projection onto \(V = \text{Col}(A)\).

Solution.

a) \(Q = \begin{pmatrix}
1/\sqrt{5} & 1/2 & 1/2 \\
-1/\sqrt{5} & 0 & 0 \\
-1/\sqrt{5} & 1/2 & 1/2 \\
1/\sqrt{5} & -1/2 & 1/2 \\
1/\sqrt{5} & 1/2 & -1/2
\end{pmatrix}\)

b) \(\hat{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\)

c) \(P_v = QQ^T = \frac{1}{10} \begin{pmatrix}
7 & -2 & 3 & 2 & 2 \\
-2 & 2 & 2 & -2 & -2 \\
3 & 2 & 7 & -2 & -2 \\
2 & -2 & 2 & 7 & -3 \\
2 & -2 & -2 & -3 & 7
\end{pmatrix}\)
Problem 3. [15 points]

Consider the data points

\[
\begin{align*}
    b_1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & b_1 &= \begin{pmatrix} 1 \\ 8 \end{pmatrix} & b_2 &= \begin{pmatrix} 3 \\ 8 \end{pmatrix} & b_3 &= \begin{pmatrix} 4 \\ 20 \end{pmatrix}
\end{align*}
\]

drawn below.

![Graph with data points](image)

a) Find the matrix \( A \) such that the least-squares solution \( \bar{x} = (C, D) \) of

\[
A \begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}
\]

gives the coefficients of the best-fit line \( y = Cx + D \).

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

c) Compute the minimized vector \( b \perp \). What does \( b \perp \) represent in the original best-fit problem? (Here \( V = \text{Col}(A) \).)

d) What is the best-fit line among all lines passing through the origin?
Solution.

a) 
\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \]

b) 
\[ \hat{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \implies y = 4x + 1 \]

c) The minimized vector is
\[ b_{\perp} = b - A\hat{x} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}. \]
This is the vector of vertical distances from the data points to the graph of the best-fit line, drawn in red in the picture.

d) Using \( y = Cx \) means solving the least-squares problem
\[ \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} C = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \implies C = \frac{56}{13}. \]
The best-fit line is \( y = \frac{56}{13}x \).
Problem 4.

A line $V$ and a vector $b$ are drawn below. Draw and label:

a) The orthogonal projection $b_V$.

b) The projection onto the orthogonal complement $b_{V\perp}$.

c) The vector $b - 2b_{V\perp}$.
Problem 5. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) \( \text{Nul} \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix} \)

b) \( \text{Col} \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 0 \\ 4 & -1 \end{pmatrix} \)

c) The subspace of all vectors in \( \mathbb{R}^4 \) whose entries sum to zero.

d) The line \( \{(t, 2t, 3t): t \in \mathbb{R}\} \).

e) \( \mathbb{R}^3 \)

Solution.

These are the bases you would obtain if you did the problem the same way I did.

\[
\begin{align*}
a) & \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix} \right\} \\
b) & \quad \left\{ \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ 0 \\ 1 \end{pmatrix} \right\} \\
c) & \quad \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\
d) & \quad \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\} \\
e) & \quad \{\} \\
\end{align*}
\]
Problem 6. [16 points]

a) Let $A$ be an $m \times n$ matrix of rank $r$. Which of the following statements are equivalent to “$A$ has full row rank”?

1. $\text{Nul}(A^T) = \{0\}$
2. $n = r$
3. $\text{Col}(A) = \mathbb{R}^m$
4. $A$ has linearly independent columns
5. $A$ has a pivot in every row
6. $A$ is invertible
7. $Ax = b$ is consistent for every vector $b$

b) Explain why the projection matrix $P_V$ onto a subspace $V$ can be written as $QQ^T$ for some matrix $Q$ with orthonormal columns. (What is $Q$ in terms of $V$?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly dependent, but $v_3$ is not in $\text{Span}\{v_1, v_2\}$. Be sure to label which is $v_3$.

d) Give an example of a $4 \times 4$ matrix $A$ such that $\text{Nul}(A) = \text{Row}(A)$, or explain why no such matrix exists.

Solution.

a) (1),(3),(5),(7)

b) Choose an orthonormal basis for $V$, then let $Q$ be the matrix with those columns.

c) There are many answers. One is

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$  

d) This is impossible since $\text{Nul}(A) = \text{Row}(A)^\perp$. 