Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!
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Problem 1. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
0 & 2 & -2 & -2 \\
2 & 0 & 8 & 0 \\
-3 & -1 & -3 & 6 \\
6 & 0 & 12 & -6
\end{pmatrix}.
\]

a) Perform Gaussian elimination with maximal partial pivoting to obtain a \( PA = LU \) decomposition of \( A \). You should end up with

\[
U = \begin{pmatrix}
6 & 0 & 12 & -6 \\
0 & 2 & -2 & -2 \\
0 & 0 & 4 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Please write the row operations you performed. (You can continue your work on the back of this sheet.)

\[
L = \begin{pmatrix}
\end{pmatrix} \quad P = \begin{pmatrix}
\end{pmatrix}
\]

b) Briefly explain the reason one might want to always choose the largest pivot in absolute value.
[Scratch work for Problem 1]
Problem 2. [20 points]

Consider the matrix

\[
A = \begin{pmatrix}
1 & 2 & -1 \\
-1 & -3 & 4 \\
-2 & -6 & 9
\end{pmatrix}.
\]

a) Compute \( A^{-1} \). Please write the row operations you performed.

\[
A^{-1} = \begin{pmatrix}
\text{\( A^{-1} = \) }
\end{pmatrix}
\]

b) Express \( A^{-1} \) as a product of elementary matrices. (Your answer will be a product of matrices with numbers in them, as opposed to row operations.)

\[
A^{-1} = \begin{pmatrix}
\text{\( A^{-1} = \) }
\end{pmatrix}
\]

c) Solve \( Ax = b \), where \( b = (b_1, b_2, b_3) \) is an unknown vector. (Your answer will be a formula in \( b_1, b_2, b_3 \).)

\[
x = \begin{pmatrix}
\text{\( x = \) }
\end{pmatrix}
\]
[Scratch work for Problem 2]
Consider the system of equations
\[ \begin{align*}
x_1 + 2x_2 - x_3 - x_4 &= 2 \\
x_2 + x_3 + 2x_4 &= 1.
\end{align*} \]

a) Express the solution set as a translate of a span:

\[
\text{solution set} = \left\{ \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \right\} + \text{Span}\left\{ \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \right\}
\]

b) The solution set is a (circle one) \( \text{point} \), \( \text{line} \), \( \text{plane} \), \( \text{space} \) in \( \mathbb{R}^4 \).

c) The solution set of \( Ax = 0 \) has dimension \( \square \).

d) Describe \( \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right\} \) geometrically:

it is a (circle one) \( \text{point} \), \( \text{line} \), \( \text{plane} \), \( \text{space} \) in \( \mathbb{R}^4 \).

e) Find numbers \( b_1, b_2 \) making the system
\[ \begin{align*}
x_1 + 2x_2 - x_3 - x_4 &= b_1 \\
x_2 + x_3 + 2x_4 &= b_2
\end{align*} \]
inconsistent. If no such numbers exist, explain why.

\[ b_1 = \square \quad b_2 = \square \]
[Scratch work for Problem 3]
Problem 4. [12 points]

Give examples of $2 \times 2$ matrices $A, B, C$ with ranks 0, 1, and 2, respectively. Draw pictures of the solution set of $Ax = 0$ and the span of the columns of $A$, and likewise for $B$ and $C$. (Recall that the rank of a matrix is the number of pivots.) Be precise!

a) Rank 0: $A = \begin{pmatrix} \ \ \ \\ \end{pmatrix}$

b) Rank 1: $B = \begin{pmatrix} \ \ \ \\ \end{pmatrix}$

c) Rank 2: $C = \begin{pmatrix} \ \ \ \\ \end{pmatrix}$

Problem 5. [10 points]

A certain $2 \times 2$ matrix $A$ has columns $v$ and $w$, pictured below. Solve the equations $Ax_1 = b_1$ and $Ax_2 = b_2$, where $b_1$ and $b_2$ are the vectors in the picture.
[Scratch work for Problems 4 and 5]
Problem 6. [16 points]

Short-answer questions: you do not need to justify your answers.

a) Suppose that $A$ is a $4 \times 2$ matrix such that the solution set of $A(x, y) = 0$ is the line $y = x$. Let $b$ be a nonzero vector in $\mathbb{R}^4$. Which of the following are definitely not the solution set of $Ax = b$? (Circle all that apply.)

The line $y = x$. The $y$-axis. The line $y = x + 1$.
The point $(1, 2, 3, 0)$. The empty set.

b) Consider the following plane in $\mathbb{R}^3$:

$$P = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$  

Find two other vectors that span $P$. Your answer cannot contain a scalar multiple of $(1, 0, -1)$ or $(1, -1, 0)$.

$$P = \text{Span}\left\{ \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \vdots \\ \vdots \end{pmatrix} \right\}$$

c) Find three vectors $u, v, w \in \mathbb{R}^3$ such that $\text{Span}\{u, v, w\}$ is a plane, but such that $w \notin \text{Span}\{u, v\}$.

$$u = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \quad v = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix} \quad w = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

d) A nonzero $2 \times 3$ matrix $A$ has the property that $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is inconsistent. The span of the columns of $A$ is a

point line plane space.
[Scratch work for Problem 6]