1. **Some simple examples**

For each of the following matrices \( A \),

i) Find the characteristic polynomial \( p(\lambda) = \det(A - \lambda I_2) \).

ii) Find all the eigenvalues by solving \( p(\lambda) = 0 \).

iii) For each eigenvalue \( \lambda_i \), find a basis of the associated eigenspace \( \text{Nul}(A - \lambda_i I_2) \).

iv) An \( n \times n \) matrix \( A \) is diagonalizable if and only if the dimensions of the eigenspaces add up to \( n \). For these matrices, you may have one or two eigenspaces, depending on how many different roots \( p(\lambda) \) has.

Is the matrix \( A \) diagonalizable? Is the matrix \( A \) diagonal?

\[
a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad b) \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad c) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad d) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
e) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad f) \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad g) \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}
\]
2. **A $2 \times 2$ diagonalization**

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$.

a) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.

b) Using the quadratic formula, find the two solutions to $p(\lambda) = 0$. The two solutions, $\lambda_1$ and $\lambda_2$, are the two eigenvalues of $A$.

c) Find the eigenvector $v_1 = (x_1, y_1)$ by solving the eigenvector equation

$$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

Note that there is more than one solution—choose any non-zero solution.

d) Find the eigenvector $v_2 = (x_2, y_2)$ by solving the eigenvector equation

$$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$$

e) Diagonalize $A$, by making a matrix of eigenvalues $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, a matrix of eigenvectors $C = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and confirming that $A = CDC^{-1}$ by multiplying these three matrices.

f) Compute the vector $A^n(1, 2)$.

**Hint:** Find scalars $c_1, c_2$ so that $(1, 2) = c_1 v_1 + c_2 v_2$. It may help to use the matrix $C^{-1}$ to do this. Then use the formula $A^n(1, 2) = c_1 A^n v_1 + c_2 A^n v_2$.

**g)** When $n$ is very large, $\|A^{n+1}(1, 2)\|/\|A^n(1, 2)\|$ is approximately _____.

**h)** When $n$ is very large, $\|A^{n+1}(1, 1)\|/\|A^n(1, 1)\|$ is approximately _____. (This should be easier than g.)

i) If you were given a random vector $w$, what would you expect $\|A^{n+1} w\|/\|A^n w\|$ to approximate when $n$ is very large?
3. **Some 3 × 3 characteristic polynomials**

Compute the characteristic polynomials and eigenvalues of the matrices

\[
A = \begin{pmatrix}
0 & 1 & -1 \\
-1 & 2 & -1 \\
-1 & 1 & 0 \\
\end{pmatrix}, \quad B = \begin{pmatrix}
-1 & 2 & -1 \\
-2 & 3 & -1 \\
-1 & 1 & 0 \\
\end{pmatrix}.
\]

Decide if each matrix is diagonalizable, and if it is, diagonalize it.
4. Traces and determinants

Recall that the trace \( \text{Tr}(A) \) is the sum of the diagonal entries of \( A \).

a) For each of the matrices in problem 1(a)–(f), factor \( p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \).
Verify that
\[
\text{Tr}(A) = \lambda_1 + \lambda_2 \quad \text{and} \quad \det(A) = \lambda_1 \cdot \lambda_2.
\]

b) For any \( n \times n \) matrix, the polynomial \( p(\lambda) = \det(A - \lambda I_n) \) can be factored as
\[
p(\lambda) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n).
\]
Verify that
\[
\det(A) = \lambda_1 \cdots \lambda_n.
\]
**Hint:** What happens to \( \det(A - \lambda I_n) \) when you set \( \lambda = 0 \)? What happens to \((-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n) \) when you set \( \lambda = 0 \)?

c) The determinant \( \det(A) \) has another product formula:
\[
\det(A) = (-1)^k d_1 \cdots d_n,
\]
when the \( A \) has REF with pivot entries \( d_1, \ldots, d_n \), found using Gaussian elimination w/o row scaling and with \( k \) row swaps. Even though this formula looks quite similar to the formula of b), eigenvalues and pivots are not at all the same.

Find an example of a \( 2 \times 2 \) matrix where the pivots \( d_1, d_2 \) are not the same as the eigenvalues \( \lambda_1, \lambda_2 \).

d) (Challenge) For any \( n \times n \) matrix, show that \( \text{Tr}(A) = \lambda_1 + \cdots + \lambda_n \).