Math 218D Problem Session

Week 8

1. Orthogonal matrices

A **orthogonal matrix** is a *square* matrix Q whose columns form an ortho*normal* set. Alternately, it is a square matrix Q such that $Q^TQ = I_n$.

a) $Q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not an orthogonal matrix, since $Q^TQ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

b) $Q = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ is not an orthogonal matrix, since $Q^TQ = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$.

c) $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an orthogonal matrix, since $Q^TQ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. Rotation and reflection

$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

a) $R_\theta^T R_\theta = \begin{pmatrix} \cos(\theta)^2 + \sin(\theta)^2 & 0 \\ 0 & \cos(\theta)^2 + \sin(\theta)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) $R_{\pi/6} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$, so $R_{\pi/6}(1,0) = (\sqrt{3}/2, 1/2)$ and $R_{\pi/6}(1,1) = (\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}+1}{2})$.

c) The angle between $(1,0)$ and $(1,1)$ is $\pi/4$. We want to confirm that the angle between $u = (\sqrt{3}/2, 1/2)$ and $v = (\frac{\sqrt{3}-1}{2}$ is $\pi/4$. We use the formula $\cos(\theta) = \frac{u \cdot v}{||u||||v||}$ (note that this is not the same θ as the variable from the rotation matrix). Compute that $||u|| = 1, ||v|| = \sqrt{2}$, and $u \cdot v = 1$. Then $\cos(\theta) = \sqrt{2}/2$, so $\theta = \pi/4$.

Consider a line $L = \text{Span}\{v\} \subset \mathbf{R}^3$, and the orthogonal complement plane $V = L^\perp$. The **reflection matrix** for **reflection across** V is the orthogonal matrix

$$Q = I_3 - 2P_L,$$

where P_L is the projection matrix for L.

d) The projection matrix is $P_L = \frac{(0,1,0)(0,1,0)^T}{(0,1,0)\cdot(0,1,0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The line L is the y-axis, and the plane V is the xz-plane. The projection of $(1,1,0)$ on to the y-axis is $(0,1,0)$. The reflection of $(1,1,0)$ across the xz-plane, $Q(1,1,0)$, is $(1,-1,0)$.

e) $Q^T Q = (I_3 - 2P_L)^T (I_3 - 2P_L) = (I_3^T - 2P_L^T)(I_3 - 2P_L) = I_3 - 2P_L^T - 2P_L + 4P_L^T P_L$. Since $P_L^T = P_L$, this becomes $I_3 - 4P_L + 4(P_L)^2$. Since $P_L^2 = P_L$, this becomes $I_3 - 4P_L + 4P_L = I_3$. Therefore $Q^T Q = I_3$.

3. Gram-Schmidt and QR

The vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are a basis for a plane $V \subset \mathbf{R}^3$. Set

$$u_1 = v_1,\ u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.$$

a) $u_1 = (1,2,-2), u_2 = (1,1,1) - \frac{(1,1,1)\cdot(1,2,-2)}{(1,2,-2)\cdot(1,2,-2)}(1,2,-2) = (1,1,1) - \frac{1}{9}(1,2,-2) = (8/9, 7/9, 11/9)$.

Then $u_1/||u_1|| = \frac{1}{3}(1,2,-2)$, and $u_2/||u_2|| = \frac{1}{\sqrt{234}}(8,7,11)$. The vectors are unit length, and $(1,2,-2)$ is orthogonal to $(8,7,11)$, so these two vectors are orthonormal.

b) $Q = \begin{pmatrix} 1/3 & 8/\sqrt{234} \\ 2/3 & 7/\sqrt{234} \\ -2/3 & 11/\sqrt{234} \end{pmatrix}$, and $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$.

$$R = Q^T A = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 8/\sqrt{234} & 7/\sqrt{234} & 11/\sqrt{234} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1/3 \\ 0 & 26/\sqrt{234} \end{pmatrix}.$$

You could note that $\sqrt{234} = 3\sqrt{26}$, to simplify this to $R = \begin{pmatrix} 3 & 1/3 \\ 0 & \sqrt{26}/3 \end{pmatrix}$.

You can check $QR = \begin{pmatrix} 1/3 & 8/\sqrt{234} \\ 2/3 & 7/\sqrt{234} \\ -2/3 & 11/\sqrt{234} \end{pmatrix} \begin{pmatrix} 3 & 1/3 \\ 0 & 26/\sqrt{234} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$ to make sure R is correct.

c) If $A = QR$, then $Q^T A = Q^T QR$. Since $Q^T Q = I$, this simplifies to $Q^T A = R$.

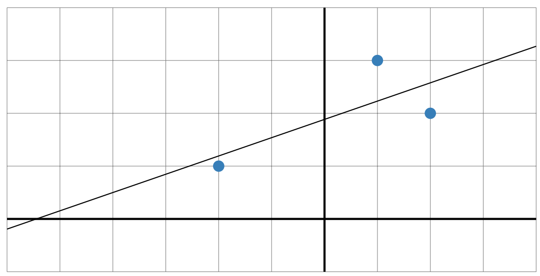
4. Least squares

$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$, the the least-squares equation is $A^T A\hat{x} = A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

a) $A^T A = \begin{pmatrix} 9 & 1 \\ 1 & 3 \end{pmatrix}$ and $A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$. The RREF of $\left(\begin{array}{cc|c} 9 & 1 & 5 \\ 1 & 3 & 6 \end{array}\right)$ is $\left(\begin{array}{cc|c} 1 & 0 & 9/26 \\ 0 & 1 & 49/26 \end{array}\right)$.

Therefore $C = 9/26, D = 49/26$.

b) We plot the data points and the least-squares line $y = \frac{9}{26}x + \frac{49}{26}$. It may help to note that this line has x-intercept $-49/9 \approx -5.44$ and y-intercept $49/26 \approx 1.88$



c) $A\hat{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 9/26 \\ 49/26 \end{pmatrix} = \begin{pmatrix} 58/26 \\ 65/26 \\ 31/26 \end{pmatrix}$. The numbers in the vector $A\hat{x}$ are the vertical distances between the data points and the best-fit line.

d) The error $||A\hat{x} - (3,2,1)||^2$ equals $||((58-78)/26, (65-52)/26, (31-26)/26)||^2 = \frac{1}{676}(20^2 + 13^2 + 5^2) = \frac{400+169+25}{676} = \frac{594}{676} \approx 0.879$.

e) We solve $R\hat{x} = Q^T(3,2,1)$, where $R = \begin{pmatrix} 3 & 1/3 \\ 0 & 26/\sqrt{234} \end{pmatrix}$ and $Q = \begin{pmatrix} 1/3 & 8/\sqrt{234} \\ 2/3 & 7/\sqrt{234} \\ -2/3 & 11/\sqrt{234} \end{pmatrix}$.

First, $Q^T(3,2,1) = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 8/\sqrt{234} & 7/\sqrt{234} & 11/\sqrt{234} \end{pmatrix}(3,2,1) = (5/3, 49/\sqrt{234})$.

We find the RREF of $\left(\begin{array}{cc|c} 3 & 1/3 & 5/3 \\ 0 & 26/\sqrt{234} & 49/\sqrt{234} \end{array}\right)$. We can immediately get rid of all the denominators by row scaling: $\left(\begin{array}{cc|c} 9 & 1 & 5 \\ 0 & 26 & 49 \end{array}\right)$. Then $\left(\begin{array}{cc|c} 9 & 0 & 81/26 \\ 0 & 26 & 49 \end{array}\right)$.

Therefore $C = 9/26, D = 49/26$.

5. **Another Gram–Schmidt**

$v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)$

a) We do Gram–Schmidt:

$$\begin{aligned}
u_1 &= (1,1,0),\\
u_2 &= (1,0,1) - \frac{(1,1,0)\cdot(1,0,1)}{(1,1,0)\cdot(1,1,0)}(1,1,0)\\
&= (1,0,1) - \frac{1}{2}(1,1,0)\\
&= (1/2,-1/2,1),\\
u_3 &= (0,1,1) - \frac{(1,1,0)\cdot(0,1,1)}{(1,1,0)\cdot(1,1,0)}(1,1,0) - \frac{(1/2,-1/2,1)\cdot(0,1,1)}{(1/2,-1/2,1)\cdot(1/2,-1/2,1)}(1/2,-1/2,1)\\
&= (0,1,1) - \frac{1}{2}(1,1,0) - \frac{1}{3}(1/2,-1/2,1)\\
&= (-2/3,2/3,2/3).
\end{aligned}$$

b) The orthonormal vectors are $\frac{1}{\sqrt{2}}(1,1,0)$, $\frac{1}{\sqrt{6}}(1,-1,2)$, $\frac{1}{\sqrt{3}}(-1,1,1)$. Therefore

$$A = \begin{pmatrix} 1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & \\ -1/\sqrt{3} & & \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}.$$

We compute

$$\begin{aligned}
R = Q^T A &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0\\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6}\\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1 \end{pmatrix}\\
&= \begin{pmatrix} 2/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 3/\sqrt{6} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}.
\end{aligned}$$

c) Consider the vector $b = (1,1,1)$. There are three equations

$$\begin{aligned}
b\cdot u_1 &= (x_1u_1 + x_2u_2 + x_3u_3)\cdot u_1,\\
b\cdot u_2 &= (x_1u_1 + x_2u_2 + x_3u_3)\cdot u_2,\\
b\cdot u_3 &= (x_1u_1 + x_2u_2 + x_3u_3)\cdot u_3.
\end{aligned}$$

This simplifies to

$$\begin{aligned}
b\cdot u_1 &= ||u_1||^2 x_1,\\
b\cdot u_2 &= ||u_2||^2 x_2,\\
b\cdot u_3 &= ||u_3||^2 x_3.,
\end{aligned}$$

since u_1, u_2, u_3 are an orthogonal set of vectors.

Recall that $u_1 = (1,1,0)$, $u_2 = (1/2,-1/2,1)$, $u_3 = (-2/3,2/3,2/3)$. We can solve for x_1, x_2, and x_3 now:

$$x_1 = \frac{b \cdot u_1}{||u_1||^2} = \frac{2}{2} = 1,$$
$$x_2 = \frac{b \cdot u_2}{||u_2||^1} = \frac{1}{(3/2)} = 2/31/3,$$
$$x_3 = \frac{b \cdot u_3}{||u_3||^2} = (2/3)/(4/3) = 1/2.$$

In other words, $(1,1,1) = (1,1,0) + \frac{2}{3}(1/2,-1/2,1) + \frac{1}{2}(-2/3,2/3,2/3)$.

d) How you could instead solve for these scalars using the formula $QQ^T = P_{\mathbf{R}^3} = I_3$? First, compute $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = Q^T b$. The formula $QQ^T = I$ implies that $Qa = Q(Q^T b) = b$. Since Q has columns $u_1/||u_1||$, $u_2/||u_2||, u_3/||u_3||$, this implies that

$$b = Qa = a_1 \frac{u_1}{||u_1||} + a_2 \frac{u_2}{||u_2||} + a_3 \frac{u_3}{||u_3||}.$$

In other words,

$$b = \frac{a_1}{||u_1||} u_1 + \frac{a_2}{||u_2||} u_2 + \frac{a_3}{||u_3||} u_3.$$

Therefore, if you compute $a = Q^T b$, and then $\frac{a_1}{||u_1||}, \frac{a_2}{||u_2||}, \frac{a_3}{||u_3||}$, you would find scalars which make b into a linear combination of u_1, u_2, and u_3.