Math 218D Problem Session

Week 7

1. Projection matrices for lines

For each of the following lines L, compute the projection matrix P_L.

a) $L = \text{Span}\{(1,1)\}$,

b) $L = \text{Span}\{(1,2,3)\}$,

c) $L = \{(x,y,z) \in \mathbf{R}^3 : 2x + y + z = 0\}^{\perp}$.

2. Projection matrices for planes

Consider the plane

$$V = \text{Span}\{(1,1,1,1),(1,2,3,4)\}$$

in $\mathbf{R}^4$.

a) Compute the projection matrix P_V for the subspace V – this is the matrix which, when multiplied with a vector b, produces the projection b_V:

$$P_V b = b_v.$$

(Feel free to use a computer to help with the matrix multiplications in the formula $P_V = A(A^TA)^{-1}A^T$ if you are finding it tedious.)

b) Compute the vectors $(I_4 - P_V)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $(I_4 - P_V)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$. Explain why these two vectors give a basis for the plane $V^{\perp}$.

c) Use your answer to **b)** to describe the plane V via *two* implicit equations:

$$V = \{(x_1,x_2,x_3,x_4) \in \mathbf{R}^4 : c_1x_1+c_2x_2+c_3x_3+c_4x_4 = 0 \text{ and } c_1'x_1+c_2'x_2+c_3'x_3+c_4'x_4 = 0\}.$$

In other words, what coefficient vectors (c_1,c_2,c_3,c_4) and (c_1',c_2',c_3',c_4') can we use to describe V, and why? Confirm that every vector in V satisfies these equations by checking that both $(1,1,1,1)$ and $(1,2,3,4)$ do.

3. **Some mistakes to avoid**

A false "fact": every projection matrix $P = A(A^TA)^{-1}A^T$ equals the identity matrix I.

A false "proof":

$$P = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I.$$

a) What is wrong would this proof?

b) In what case would this proof be correct?

Consider the subspace $V = \text{Span}\{(1,1,1,-1),\ (2,1,1,2),\ (3,2,2,1)\}$ in $\mathbb{R}^4$. V is the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) It would be *incorrect* to say that $P = A(A^TA)^{-1}A^T$ is the projection matrix for V. Why?

Hint: Try computing P - what goes wrong?

d) Find a matrix B so that $P = B(B^TB)^{-1}B^T$ is the projection matrix for V – you do not need to compute B.