1. **Projection matrices for lines**

   For each of the following lines $L$, compute the projection matrix $P_L$.

   a) $L = \text{Span}\{(1, 1)\}$,

   b) $L = \text{Span}\{(1, 2, 3)\}$,

   c) $L = \{(x, y, z) \in \mathbb{R}^3 : 2x + y + z = 0\}^\perp$. 
2. **Projection matrices for planes**

Consider the plane

\[ V = \text{Span}\{(1, 1, 1, 1), (1, 2, 3, 4)\} \]

in \( \mathbb{R}^4 \).

a) Compute the projection matrix \( P_V \) for the subspace \( V \) – this is the matrix which, when multiplied with a vector \( b \), produces the projection \( b_V \):

\[ P_V b = b_V. \]

(Feel free to use a computer to help with the matrix multiplications in the formula \( P_V = A(A^T A)^{-1}A^T \) if you are finding it tedious.)

b) Compute the vectors \((I_4 - P_V)\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\) and \((I_4 - P_V)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\). Explain why these two vectors give a basis for the plane \( V^\perp \).

c) Use your answer to b) to describe the plane \( V \) via two implicit equations:

\[ V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0 \text{ and } c_1' x_1 + c_2' x_2 + c_3' x_3 + c_4' x_4 = 0\}. \]

In other words, what coefficient vectors \((c_1, c_2, c_3, c_4)\) and \((c_1', c_2', c_3', c_4')\) can we use to describe \( V \), and why? Confirm that every vector in \( V \) satisfies these equations by checking that both \((1, 1, 1, 1)\) and \((1, 2, 3, 4)\) do.
3. Some mistakes to avoid

A false “fact”: every projection matrix \( P = A(A^T A)^{-1} A^T \) equals the identity matrix \( I \).

A false “proof”:

\[
P = A(A^T A)^{-1} A^T = AA^{-1}(A^T)^{-1}A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I.
\]

a) What is wrong with this proof?

b) In what case would this proof be correct?

Consider the subspace \( V = \text{Span}\{(1, 1, 1, -1), (2, 1, 1, 2), (3, 2, 2, 1)\} \) in \( \mathbb{R}^4 \). \( V \) is the column space of the matrix

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
1 & 1 & 2 \\
1 & 1 & 2 \\
-1 & 2 & 1
\end{pmatrix}.
\]

c) It would be incorrect to say that \( P = A(A^T A)^{-1} A^T \) is the projection matrix for \( V \). Why?

Hint: Try computing \( P \) - what goes wrong?

d) Find a matrix \( B \) so that \( P = B(B^T B)^{-1} B^T \) is the projection matrix for \( V \) – you do not need to compute \( B \).