Math 218D Problem Session  
Week 4

1. Parallel lines
   a) Span\{(−2, 1, 1)\} + (3, −2, 1)
   b) One possible answer is \(P_1 = (3, −2, 1), P_2 = (1, −1, 2), P_2 - P_1 = −(−2, 1, 1)\).
   c) To get a parallel line, you need the same matrix \(A\) but a different \(b\) vector. You can find the correct \(b\) vector by multiplying \(A\) times \(x = (1, 1, 1): A(1, 1, 1) = (3, 6)\). In other words, the solution set of\[
x + y + z = 3 \\
2x + 3y + z = 6
\]
is parallel to \(L\) and passing through \((1, 1, 1)\).

2. The geometry of spans
   a) No, it is not possible. You can confirm this by computing the RREF of \[
\begin{pmatrix}
1 & 1 & 1 \\
−1 & −1 & 1 \\
5 & 1 & 0
\end{pmatrix}
\].

   Alternately, you could observe that that the first two components of \[
\begin{pmatrix}1 \\
−1 \\
5
\end{pmatrix}
\]and \[
\begin{pmatrix}−1 \\
1 \\
1
\end{pmatrix}
\]add up to 0, while the first two components of \[
\begin{pmatrix}1 \\
0
\end{pmatrix}
\]do not.
   b) It is all of \(\mathbb{R}^3\), since \[
\begin{pmatrix}1 \\
1 \\
0
\end{pmatrix}
\]is not contained in the plane Span \[
\{(−1, 1, 1), (−1, −1, 1)\}
\] (using 3a)).
   c) By computing the REF of \[
\begin{pmatrix}
1 & 1 & b_1 \\
−1 & −1 & b_2 \\
5 & 1 & b_3
\end{pmatrix}
\], we confirm that the vectors \(b = (b_1, b_2, b_3)\) which make
\[
\begin{pmatrix}
1 & 1 \\
−1 & −1 \\
5 & 1
\end{pmatrix}
\]
consistent are precisely those where \(b_1 + b_2 = 0\). This means that the plane parametrized by
\[
x_1\begin{pmatrix}1 \\
−1 \\
5
\end{pmatrix} + x_2\begin{pmatrix}−1 \\
−1 \\
1
\end{pmatrix}
\]
has equation \(x + y = 0\).
d) Yes, you can find scalars \( x_1, x_2 \) so that
\[
\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},
\]
since \((4, -4, 0)\) solves the equation \(x + y = 0\) found in 3c).

e) The vectors \( \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \) are not parallel, so they span a plane. The third vector \( \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix} \) is contained in that plane by 3d), so adding it to the list of vectors does not enlarge the span.

3. Subspaces?

a) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

b) A subspace, since it is the solution set of a homogeneous linear equation.

c) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

d) A subspace, since it is the left-null space of the matrix \( \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \)

e) Not a subspace, since it doesn’t contain \((0, 0, 0)\).

f) A subspace, since \( \{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + y^2 = 0\} = \{(x, y) \in \mathbb{R}^2 : (x+y)^2 = 0\} = \{(x, y) \in \mathbb{R}^2 : x+y = 0\} \).

4. The fundamental subspaces I

a) The \( \text{Nul}(A) \) and \( \text{Nul}(A^T) \) are points, while the \( \text{Col}(A) \) and \( \text{Col}(A^T) \) are all of \( \mathbb{R}^2 \).

b) \( \text{dim}(\text{Nul}(A)) + \text{dim}(\text{Col}(A^T)) = 2 \)

5. The fundamental subspaces II

a) The spanning sets are \( \text{Col}(A^T) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \), \( \text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\} \), \( \text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \), \( \text{Nul}(A^T) = \text{Span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\} \), although other answers are possible.

b) Draw the lines spanned by the vectors of a).

c) \( \text{dim}(\text{Nul}(A)) + \text{dim}(\text{Col}(A^T)) = 2 \).

d) The lines \( \text{Nul}(A) \) and \( \text{Col}(A^T) \) are perpendicular. The lines \( \text{Col}(A) \) and \( \text{Nul}(A^T) \) are perpendicular.

6. The fundamental subspaces III
a) $\text{Col}(A^T)$ is a subspace of $\mathbb{R}^3$

b) $\text{Nul}(A)$ is a subspace of $\mathbb{R}^3$

c) $\text{Col}(A)$ is a subspace of $\mathbb{R}^2$

d) $\text{Nul}(A^T)$ is a subspace of $\mathbb{R}^2$

e) The column space is the line spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and the left-null space is the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

f) The subspace $\text{Col}(A^T)$ is spanned by the vectors $(1,-1,2)$ and $(-2,2,-4)$, but these are scalar multiples of each other, so the row space is a line. The null space can be found via RREF: $\text{rref}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. The free variables are $y$ and $z$, and the parametric form is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Therefore the null space is a plane in $\mathbb{R}^3$.

g) $\text{Col}(A^T) = \text{Span}\{(1,-1,2)\}$

h) $\text{Nul}(A) = \text{Span}\{(1,1,0),(-2,0,1)\}$

i) We consider the matrix $B = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, whose column space equals $\text{Nul}(A)$. We find an equation for the column space of $B$ by finding the RREF of $\begin{pmatrix} 1 & -2 & b_1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{pmatrix}$, and finding the equation which makes the system consistent. The RREF of this augmented matrix is

$\begin{pmatrix} 1 & -2 & b_1 \\ 0 & 2 & b_2 - b_1 \\ 0 & 0 & b_1 - b_2 + 2b_3 \end{pmatrix}$.

The equation that $(b_1, b_2, b_3)$ must satisfy to be in the column space of $B$ (and hence the null space of $A$) is $b_1 - b_2 + 2b_3 = 0$. In other words, the equation for the plane $\text{Nul}(A)$ is $x - y + 2z = 0$.

j) The coefficients of the equation are $(1,-1,2)$. This is the same as the vector which spanned $\text{Col}(A^T)$ (you may have gotten a scalar multiple of the vector spanning $\text{Col}(A^T)$ instead.) The means that every vector in the plane is perpendicular to the vector $(1,-1,2)$, i.e. that the plane has normal vector $(1,-1,2)$. In other words, the null space is orthogonal to the row space. We will discuss the orthogonality of subspaces in more detail in Week 6.