Math 218D Problem Session

Week 4

1. Parallel lines

The solution set of

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + z &= 0 \end{aligned}$$

is a line L inside of $\mathbf{R}^3$.

a) Describe the line L as the translate of a span, i.e. as

$$\text{Span}\{(v_1, v_2, v_3)\} + (c_1, c_2, c_3).$$

What point does the line L pass through?

b) Find two different points, P_1 and P_2, on L. Verify that $P_2 - P_1$ is contained in the span $\text{Span}\{(v_1, v_2, v_3)\}$ you found in part **a)**.

c) Find a different system of two linear equations whose solution set is *parallel* to L, passing through the point $(1, 1, 1)$.

2. The geometry of spans

a) Is it possible to find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

Solve the system algebraically, then geometrically using this demo.

b) Describe

$$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.

c) Find an equation $ax+by+cz=d$ (a, b, c, d scalars) for the plane parametrized by

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

as we vary x_1 and x_2.

Hint: Describe all the vectors $b = (b_1, b_2, b_3)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

consistent.

d) It is possible to find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}?$$

Explain why, *without* finding x_1 and x_2. Then find x_1 and x_2 using this demo.

e) Describe the span of the vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$. Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.

3. Subspaces?

Decide if each of the following sets of vectors is or is not a subspace, and explain why or why not.

a)

$$\{(x, y, z) \in \mathbf{R}^3 \mid x + y = 1 - z\}$$

b)

$$\{(x, y) \in \mathbf{R}^2 \mid x - 2y = 0\}$$

c) For A a 3×3 matrix, the set

$$\left\{ v \in \mathbf{R}^3 \mid Av = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) The set

$$\left\{ (x, y) \in \mathbf{R}^2 \mid \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\}$$

e)

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

f)

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + 2xy + y^2 = 0\}$$

The 4 *fundamental subspaces* associated to a matrix A are Nul(A), Col(A), Nul(A^T), and Col(A^T).

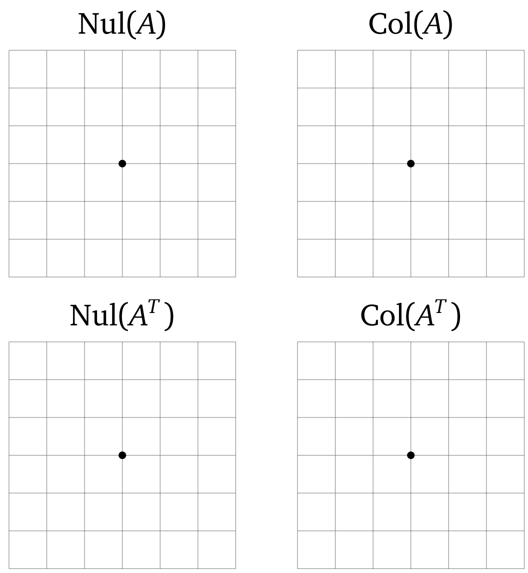
A spanning set for the null space Nul(A) can be found by finding the parametrized vector form of the solution set of $Ax = 0$.

4. The fundamental subspaces I

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

a) Find a spanning set for each of the four fundamental subspaces of this matrix A.

b) Draw each of the fundamental subspaces:



c) Compute dim(Nul(A))+dim(Col(A)), where dim refers to the *dimension* of the subspace. The dimension of a point, line, or plane is 0, 1, or 2.

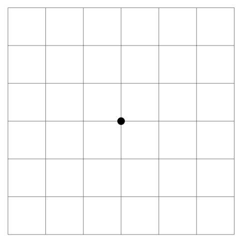
5. The fundamental subspaces II

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

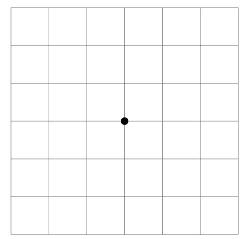
a) Find a spanning set for each of the four fundamental subspaces of the matrix A.

b) Draw each of the fundamental subspaces:

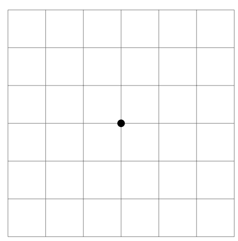
Nul(A)



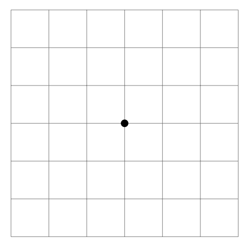
Col(A)



Nul(A^T)



Col(A^T)



c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

d) Describe a geometric relationship between Nul(A) and Col(A^T). Then describe the relationship between Col(A) and Nul(A^T).

6. The fundamental subspaces III

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Is $\mathrm{Col}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

b) Is $\mathrm{Nul}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

c) Is $\mathrm{Col}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

d) Is $\mathrm{Nul}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

e) Two of the four subspaces are contained in $\mathbf{R}^2$. Draw these two subspaces, and describe the geometric relationship between them.

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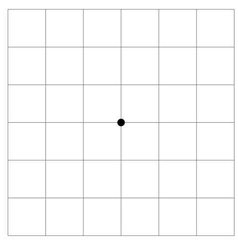

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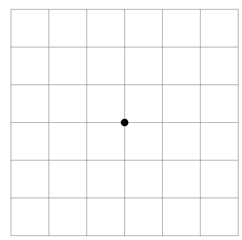

f) Two of the four subspaces are contained in $\mathbf{R}^3$. For this matrix, one is a line and the other is a plane. Determine which is which.

g) Find a vector whose span is the line.

h) Find two vectors whose span is the plane.

i) Find an equation $a_1x + a_2y + a_3z = 0$ for the plane.

Hint: Make the two vectors from **h)** into the columns of a matrix B. Find an equation which b_1, b_2, b_3 must satisfy in order for $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to be consistent. Why does this answer the question?

j) What can you observe about the relationship between the answers to **g)** and **i)**? What does this mean geometrically?