Math 218D Problem Session

Week 3 Solutions

1. Finding $PA = LU$

a) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$.

b) $L. = \begin{pmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -20 \\ 0 & 0 & -15 \end{pmatrix}$.

c) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.1 & -0.2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{pmatrix}$.

2. Solving $Ax = b$ using $PA = LU$

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$.

b) $Pb = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$. P swaps the first and second raws, and then swap the second and third rows of b.

c)

$$\begin{aligned} c_1 \qquad\qquad &= 5 \\ \tfrac{1}{2}c_1 \quad + c_2 \qquad &= 2 \\ \tfrac{1}{2}c_1 + 0c_2 + c_3 &= 1 \end{aligned},$$

and substitution gives $(c_1, c_2, c_3) = (5, -\frac{1}{2}, -\frac{3}{2})$.

d)

$$\begin{aligned} 2x_1 + 2x_2 + 5x_3 &= 5 \\ x_2 + \tfrac{1}{2}x_3 &= -\tfrac{1}{2} \\ -\tfrac{3}{2}x_3 &= -\tfrac{3}{2} \end{aligned},$$

and substitution gives $(x_1, x_2, x_3) = (1, -1, 1)$.

e) Check $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$.

3. Parametric forms

a) The RREF is $\left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) The parametric form of the solution is:

$$\begin{aligned} x_1 &= -\tfrac{1}{3}x_3 + \tfrac{1}{3} \\ x_2 &= -\tfrac{2}{3}x_2 - \tfrac{1}{3} \\ x_3 &= \ x_3 \end{aligned} \ .$$

Setting $x_3 = 0$ gives one solution: $(x_1, x_2, x_3) = (1/3, -1/3, 0)$. Setting $x_3 = 1$ gives another solution $(x_1, x_2, x_3 = (0, -1, 1)$.

c) The parametric vector form is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}.$$

d) The line passes through the point $(1/3, -1/3, 0)$, and goes in the direction of the vector $(-1/3, -2/3, 1)$.

e) This system of equations has no solutions.

f) The parametric vector form of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix}.$$

The solution to the homogeneous equation is a line, parallel to the line from part d), passing through the origin.

g) A vector $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ makes $Ax = b$ consistent precisely when $-b_1 + b_2 - b_3 = 0$.

h) The span of these vectors is the same as the set of vectors making $Ax = b$ consistent. By g), this is the same as the vectors which satisfying a single linear equation. The set of vectors satisfying a single linear equation is a plane.