Math 218D Problem Session

Week 2

1. Reduced Row Echelon Form

For each of the following augmented matrices:

a) $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right)$,

b) $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$,

c) $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right)$,

d) $\left(\begin{array}{c|c} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{array}\right)$,

answer the following questions:

(1) determine whether or not it is in REF;
(2) determine whether or not it is in RREF;
(3) if it is not in RREF, use the Gauss–Jordan algorithm to convert it into RREF;
(4) circle the pivots of the augmented matrices and determine the number of solutions to the corresponding system of linear equations.

2. Elementary matrices

a) Write down the 3×3 elementary matrix which E_1 which performs the row operation $R_2 \mathrel{+}= 3R_1$.

b) Write down the 3×3 elementary matrix E_2 which performs the row operation $R_1 \longleftrightarrow R_3$.

c) What matrix performs the (non-elementary) row operation that first does $R_2 \mathrel{+}= 3R_1$, and *then* does $R_1 \longleftrightarrow R_3$?

3. True or false?

If true, give an explanation; if false, give a counterexample.

a) All elementary matrices are invertible.

b) A 3×2 matrix can have 3 pivots.

c) If a 3×4 augmented matrix has 3 pivots, then the corresponding equations have a unique solution.

4. **Solving $Ax = b$ using $A = LU$**

When you know the LU factorization of a matrix A, you can use it to solve the matrix equation $Ax = b$. In this problem we will go through this process in an example.

Solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

using the $A = LU$ decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Identify A, b, L, and U.

b) Convert $Lc = b$ into 3 linear equations, and solve for $c = (c_1, c_2, c_3)$ using forward-substitution.

c) Convert $Ux = c$ into 3 linear equations, and solve for x using back-substitution.

d) Check your answer, by multiplying $A \cdot x$ and confirming that it equals b.

Why does this work? If $Lc = b$ and $c = Ux$, then $L(Ux) = b$. Since $A = LU$, this means that $Ax = b$.

5. Finding $A = LU$ and A^{-1} using elementary matrices

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 4 \\ 1 & 4 & 6 \end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF (not RREF) using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\qquad\qquad} A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \underline{\qquad\qquad} U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

e) Explain how to reduce U to the 3×3 identity matrix using three more elementary matrices E_4, E_5, E_6 (scaling, followed by row replacements).

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \underline{\qquad\qquad\qquad}.$$

g) Compute A^{-1} by row reducing $\left(A \mid I_n\right)$. This is exactly the same as evaluating the product above!