Math 218D Problem Session

Week 12

1. Some quick matrix exponentials

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, $e^A = \begin{pmatrix} e^2 & 0 \\ 0 & e^{-3} \end{pmatrix}$.

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. A has diagonalization $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{-1}$, so $e^A = e^{CDC^{-1}} = Ce^D C^{-1} = C\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, which you can multiply to get the final answer.

(3) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e^A = I + A + A^2/2 + \cdots = I + A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $e^A = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \cdot e^{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & 2e \\ 0 & e \end{pmatrix}$.

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, $e^A = e^{3I} \cdot e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} = e^3(I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2/2) = \begin{pmatrix} e^3 & e^3 & e^3/2 \\ 0 & e^3 & e^3 \\ 0 & 0 & e^3 \end{pmatrix}$.

2. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(y)$$

a) The matrix A in the matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

is $A = \begin{pmatrix} 3 & 2 \\ 4 & -4 \end{pmatrix}$.

b) This matrix A has characteristic polynomial $\lambda^2 + \lambda - 20$, with eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 4$. The eigenvectors are $w_1 = (-2, 8)$ and $w_2 = (-1, 2)$.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, your scalars must solve $(1, 1) = a_1 w_1 + a_2 w_2$. You can solve this by solving the system of linear equations $\left(\begin{array}{cc|c} -2 & -1 & 1 \\ 8 & 2 & 1 \end{array}\right)$. This has solution $a_1 = -5/4, a_2 = 3/2$, i.e. $-5/4(-2, 8) + 3/2(-1, 2) = (1, 1)$.

d) The solution is $u(t) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. When we plug this into the differential equation, we get $u'(t) = a_1 \lambda_1 e^{\lambda_1 t} w_1 + a_2 \lambda_2 e^{\lambda_2 t} w_2$ on one side, and $Au(t) = a_1 e^{\lambda_1 t}(\lambda_1 w_1) + a_2 \lambda_2 e^{\lambda_2 t}(\lambda_2 w_2)$ on the other. Since these are equal, $u(t)$ solves the differential equation. (We didn't actually need to use the values for a_1 and a_2 to check this.)

e) The value of $(x(1), y(1))$ is $-5/4 e^{\lambda_1} w_1 + 3/2 e^{\lambda_2} w_2 = (5/2 e^{-5} - 3/2 e^4, -10 e^{-5} + 3e^4)$. You don't need to simplify any further than this.

3. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) The matrix differential equation would be

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

b) The characteristic polynomial is $(\lambda - 1)^2 + 1 = 0$, which gives the eigenvalues $\lambda_1 = 1 + i$ and $\lambda_1 = 1 - i$. The associated eigenvectors are $w_1 = (i, 1)$ and $w_2 = (-i, 1)$.

c) The eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$ is

$$e^{t(1+i)} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

from which we can observe the real and imaginary part.

d) We write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Hence the solution would be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{i}{2} e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} e^{(1-i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

4. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x,y) = 1$ using a computer, and describe the shape (for example, for **a)** you should get an ellipse in $\mathbf{R}^2$, not an elliptic paraboloid in $\mathbf{R}^3$).

(2) Find the 2×2 symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x,y) = 2x^2 + 3y^2$ has $S = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, which is positive-definite, and $q = 1$ is an ellipse.

b) $q(x,y) = x^2 - 5y^2$ has $S = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

c) $q(x,y) = y^2$ has $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two lines.

d) $q(x,y) = -3x^2 - 2y^2$ has $S = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$, is not positive-definite, and $q = 1$ is empty.

e) $q(x,y) = x^2 + 3xy + y^2$ has $S = \begin{pmatrix} 1 & 3/2 \\ 3/2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

f) $q(x,y) = 2x^2 + 4xy + y^2$ has $S = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

g) $q(x,y) = x^2 - 4xy + 5y^2$ has $S = \begin{pmatrix} 1 & -2 \\ -2 & \\ 5 & \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipse.

h) $q(x,y,z) = x^2 + y^2 + z^2 + xy + yz + xz$ has $S = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipsoid.

i) $q(x,y,z) = x^2+y^2+z^2+2xy+2yz+2xz$ has $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two planes.