Math 218D Problem Session

Week 12

1. Some quick matrix exponentials

Compute the matrix exponential e^A of:

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$,

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,

(3) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

2. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(t)$$

a) Write this as a matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

What is the matrix A?

b) For this matrix A, find the eigenvalues λ_1 and λ_2, as well as the eigenvectors w_1 and w_2.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, which scalars a_1 and a_2 should you choose?

d) Plug the solution with initial value $(x(0), y(0)) = (1, 1)$ to the differential equation, and confirm that it is a solution.

e) For the solution you found in **c)**, compute $(x(1), y(1))$.

3. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) Write this as a matrix differential equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

b) Compute the eigenvalues $\lambda_1,\ \lambda_2$ and eigenvectors $v_1,\ v_2$ of the matrix A.

c) Compute the real and imaginary parts of the "eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$. This gives you two different *real* solutions to the differential equation.

d) Find the solution $(x(t), y(t))$ with initial value $(x(0), y(0)) = (1, 0)$.

4. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x, y) = 1$ using a computer, and describe the shape (for example, for **a)** you should get an ellipse in $\mathbf{R}^2$, not an elliptic paraboloid in $\mathbf{R}^3$).

(2) Find the 2×2 symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x, y) = 2x^2 + 3y^2$

b) $q(x, y) = x^2 - 5y^2$

c) $q(x, y) = y^2$

d) $q(x, y) = -3x^2 - 2y^2$

e) $q(x, y) = x^2 + 3xy + y^2$

f) $q(x, y) = 2x^2 + 4xy + y^2$

g) $q(x, y) = x^2 - 4xy + 5y^2$

Our final two quadratic forms are in 3 variables: this means that S is a 3×3 matrix, and $q(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

h) $q(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz$

i) $q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$