1. **Some simple examples**  
For each of the following matrices $A$,

i) Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.

ii) Find all the eigenvalues by solving $p(\lambda) = 0$.

iii) For each eigenvalue $\lambda_i$, find a basis of the associated eigenspace $\text{Nul}(A - \lambda_i I_2)$.

iv) An $n \times n$ matrix $A$ is diagonalizable if and only if the dimensions of the eigenspaces add up to $n$. For these matrices, you may have one or two eigenspaces, depending on how many different roots $p(\lambda)$ has.

Is the matrix $A$ diagonalizable? Is the matrix $A$ diagonal?

\[
\begin{align*}
\text{a)} & \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{b)} & \quad \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} & \text{c)} & \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{d)} & \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\text{e)} & \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{f)} & \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} & \text{g)} & \quad \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}
\end{align*}
\]
2. **A $2 \times 2$ diagonalization**

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$.

a) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.

b) Using the quadratic formula, find the two solutions to $p(\lambda) = 0$. The two solutions, $\lambda_1$ and $\lambda_2$, are the two eigenvalues of $A$.

c) Find the eigenvector $v_1 = (x_1, y_1)$ by solving the eigenvector equation
\[(A - \lambda_1 I_2)v_1 = 0\]

Note that there is more than one solution—choose any non-zero solution.

d) Find the eigenvector $v_2 = (x_2, y_2)$ by solving the eigenvector equation
\[(A - \lambda_2 I_2)v_2 = 0\]

e) Diagonalize $A$, by making a matrix of eigenvalues $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, a matrix of eigenvectors $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, and confirming that $A = CDC^{-1}$ by multiplying these three matrices.

f) Compute the vector $A^n(1, 2)$.

**Hint:** Find scalars $c_1, c_2$ so that $(1, 2) = c_1 v_1 + c_2 v_2$. It may help to use the matrix $C^{-1}$ to do this. Then use the formula $A^n(c_1 v_1 + c_2 v_2) = c_1 A^n v_1 + c_2 A^n v_2$.

g) When $n$ is very large, $\|A^{n+1}(1, 2)\|/\|A^n(1, 2)\|$ is approximately _____.

h) When $n$ is very large, $\|A^{n+1}(1, 1)\|/\|A^n(1, 1)\|$ is approximately _____. (This should be easier than g.)

i) If you were given a random vector $w$, what would you expect $\|A^{n+1}w\|/\|A^n w\|$ to approximate when $n$ is very large?
3. **Some $3 \times 3$ characteristic polynomials**

Compute the characteristic polynomials and eigenvalues of the matrices

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$ 

Decide if each matrix is diagonalizable, and if it is, diagonalize it.
4. **Traces and determinants**

Recall that the trace $\text{Tr}(A)$ is the sum of the diagonal entries of $A$.

**a)** For each of the matrices in problem 1(a)–(f), factor $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Verify that

$$\text{Tr}(A) = \lambda_1 + \lambda_2 \text{ and } \det(A) = \lambda_1 \cdot \lambda_2.$$ **b)** For any $n \times n$ matrix, the polynomial $p(\lambda) = \det(A - \lambda I_n)$ can be factored as

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n).$$ Verify that

$$\det(A) = \lambda_1 \cdots \lambda_n.$$ **Hint:** What happens to $\det(A - \lambda I_n)$ when you set $\lambda = 0$? What happens to $(-1)^n(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$ when you set $\lambda = 0$?

**c)** The determinant $\det(A)$ has another product formula:

$$\det(A) = (-1)^kd_1 \cdots d_n,$$

when the $A$ has REF with pivot entries $d_1, \ldots, d_n$, found using Gaussian elimination w/o row scaling and with $k$ row swaps. Even though this formula looks quite similar to the formula of **b)**, eigenvalues and pivots are not at all the same.

Find an example of a $2 \times 2$ matrix where the pivots $d_1, d_2$ are not the same as the eigenvalues $\lambda_1, \lambda_2$.

**d)** *(Challenge)* For any $n \times n$ matrix, show that $\text{Tr}(A) = \lambda_1 + \cdots + \lambda_n$. 