The Four Subspaces

Recall: To any matrix $A$, we can associate:

- $\text{Col}(A)$; basis = pivot columns of $A$
- $\text{Nul}(A)$; basis = vectors in the PVF of $A\mathbf{x} = \mathbf{0}$

There are two more subspaces: just replace $A$ by $A^T$, then take $\text{Col}$ & $\text{Nul}$.

Why? Next week...

Def: The row space of $A$ is $\text{Row}(A) = \text{Col}(A^T)$.

This is the subspace spanned by the rows of $A$, regarded as vectors in $\mathbb{R}^n$.

This is a subspace of $\mathbb{R}^n$, $n = \# \text{columns}$

($n = \# \text{entries in each row}$)

\[ \rightarrow \text{row picture} \]

Eg: $\text{Row} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1/3 \\ 1/6 \\ 1/9 \end{pmatrix}, \begin{pmatrix} 4/3 \\ 2/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 7/3 \\ 1/3 \\ 0 \end{pmatrix} \right\}$

\[ = \text{Col} \begin{pmatrix} 1/3 \\ 1/6 \\ 1/9 \end{pmatrix} \]

Fact: Row operations do not change the row space.
Why? If the rows are $v_1, v_2, v_3$, then $\text{Row}(A) = \text{Span}\{v_1, v_2, v_3\}$. Row ops:
- $R_i \leftarrow R_3$: $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_3, v_2, v_3\}$
- $R_i \times 3$: $\text{Span}\{v_1, v_3, v_3\} = \text{Span}\{v_1, 3v_2, v_3\}$
- $R_i + 2R_1$: $\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2 + 2v_1, v_3\}$

because $v_2 + 2v_1 \in \text{Span}\{v_1, v_2, v_3\}$
and $v_2 = (v_2 + 2v_1) - 2v_1 \in \text{Span}\{v_1, v_2 + 2v_1, v_3\}$

This is a col space (of $A^T$), so you know how to compute a basis (pivot columns of $A^T$). But you can also find a basis by doing elimination on $A$:

Thm: The nonzero rows of a REF of $A$ form a basis for $\text{Row}(A)$.

Eg:

$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \right\}$
Proof: by 'forward-substitution':

1. Span: row ops don't change Row(A), and you can always delete the zero vector without changing the span

\[
0 = x_1 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3x_2 \\ -3x_2 \end{pmatrix}
\]

\[x_1 = \text{pivot}, \text{ so this entry in the sum is just (1) } x_1 = 0 \Rightarrow x_1 = 0\]

\[x_2 = \text{pivot}, \text{ so this entry in the sum is just (3) } x_2 = 0 \Rightarrow x_2 = 0\]

Consequence: \(\text{dim Row}(A) = \# \text{ pivot rows} = \# \text{ pivots} = \text{rank}\).

(a nonzero row of an REF matrix has a pivot)

Def: The left null space of A is \(\text{Null}(A^T)\). This is the solution set of \(A^T x = 0\).

Notation: Just \(\text{Null}(A^T)\) (no new notation)

This is a subspace of \(\mathbb{R}^m\), \(m = \#\text{rows}\) \(\Rightarrow \text{column picture}\)
\[
A^T x = 0 \iff 0 = (A^T)^T = x^T A
\]
so \( \text{Null}(A^T) = \{\text{row vectors } y \in \mathbb{R}^m: yA = 0\} \)

\( \text{Null}(A^T) \) is a null space, so you know how to compute a basis (PFE of \( A^T x = 0 \)). You can also find a basis by doing elimination on \( A^T \):

**Thm:** If \( EA = U \) for \( E \) an invertible \( m \times m \) matrix and \( U \) a matrix in REF, and if \( U \) has \( m-r \) zero rows, then the last \( m-r \) rows of \( E \) form a basis for \( \text{Null}(A^T) \).

**Consequence:** \( \dim \text{Null}(A^T) = m-r = \# \text{rows} - \text{rank} \)

Where did \( E \) come from? Elementary matrices! Doing row ops means left-multiplying by those:

\[
A \rightarrow E_1 A \rightarrow E_2 E_1 A \rightarrow E_3 E_2 E_1 A = U
\]

so \( EA = U \) for \( E = E_3 E_2 E_1 \), which is the matrix you get by doing the same row ops on \( I_m \).
Procedure: To compute a basis of $\text{Nul}(A^T)$:

1. Form the augmented matrix $(A \mid \mathbf{I}_m)$
2. Eliminate to REF
3. The rows on the right side of the line corresponding to zero rows on the left form a basis of $\text{Nul}(A^T)$.

Eg: $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix}$

\[
\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -3 & -3 \end{bmatrix}
\]

Basis for $\text{Nul}(A^T)$: $\{(1)\}$

Check: $(1 \ -1 \ -1) \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = (0 \ 0 \ 0)$
Proof of the Thm: If \( U \) is in \( \text{REF} \) and the last \( m-r \) cols are zero then

\[
\text{Null}(U^T) = \text{Span}\{e_{m-r+1}, e_{m-r+2}, \ldots, e_m\}
\]

This is because \( U^Te_i = \) the \( i^{th} \) row of \( U \) we know from before that the nonzero rows of \( U \) are LI. And \( U^Te_{m-r+i} = \) a zero row, so \( e_{m-r+i} \in \text{Null}(U^T) \).

But \( U = EA \), so \( U^T = A^TE^T \), and

\[
0 = U^Te_{m-r+i} = A^TE^Te_{m-r+i} = A^T(m-r+i^{th} \text{ row of } E).
\]

NB: The left null space is changed by row operations:

\[
A = \begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 4 & 1 & -1 \\
1 & 2 & -1 & -2
\end{bmatrix}
\]

\[
\text{Null}(A^T) = \text{Span}\{v(1,1)\}
\]

\[
\tilde{v}
\]

\[
U = \begin{bmatrix}
1 & 2 & 2 & 1 \\
0 & 0 & -3 & 1/3
\end{bmatrix}
\]

\[
\text{Null}(U^T) = \text{Span}\{v(0,1)\}
\]
### Summary: Four Subspaces

$A$ is an $m \times n$ matrix of rank $r$

<table>
<thead>
<tr>
<th>Subspace</th>
<th>of</th>
<th>row/col</th>
<th>dim</th>
<th>basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Col}(A)$</td>
<td>$\mathbb{R}^m$</td>
<td>col</td>
<td>$r$</td>
<td>pivot cols of $A$</td>
</tr>
<tr>
<td>$\text{Nul}(A)$</td>
<td>$\mathbb{R}^n$</td>
<td>row</td>
<td>$n-r$</td>
<td>vectors in PVF</td>
</tr>
<tr>
<td>$\text{Row}(A)$</td>
<td>$\mathbb{R}^n$</td>
<td>row</td>
<td>$r$</td>
<td>nonzero rows of $REF$</td>
</tr>
<tr>
<td>$\text{Nul}(A^T)$</td>
<td>$\mathbb{R}^m$</td>
<td>col</td>
<td>$m-r$</td>
<td>last $m-r$ rows of $E$</td>
</tr>
</tbody>
</table>

The **row picture** subspaces ($\text{Nul}(A)$, $\text{Row}(A)$) are unchanged by row operations.

The **col picture** subspaces ($\text{Col}(A)$, $\text{Nul}(A^T)$) are changed by row operations.
Consequences:

Row Rank = Column Rank
\[ \dim \text{Row}(A) = \text{rank} = \dim \text{Col}(A) \]

So \( A \) & \( A^T \) have the same \# pivots — in completely different positions! (HW#4.19)

Rank - Nullity
\[ \dim \text{Col}(A) + \dim \text{Null}(A) = n = \# \text{cols} \]
\[ \dim \text{Row}(A) + \dim \text{Null}(A^T) = m = \# \text{rows} \]

NB: You can compute bases for all four subspaces by doing elimination once.

\[ A \rightarrow [A \mid \text{Im}] \rightarrow [\text{RREF}(A) \mid E] \]

- Get the pivots of \( A \rightarrow \text{Col}(A) \)
- Get \( \text{RREF}(A) \rightarrow \text{PVF of } Ax=0 \rightarrow \text{Null}(A) \)
- Get nonzero rows of \( \text{RREF}(A) \rightarrow \text{Row}(A) \)
- Get rows of \( E \rightarrow \text{Null}(A^T) \)
Full-Rank Matrices

Now we can see how the rank of a matrix can control its properties.

Def: An $m \times n$ matrix $A$ of rank $r$ has:
- full column rank if $r = n$
- full row rank if $r = m$

Thm: The Following Are Equivalent
(all are true for a given matrix $A$, or all false)

1. $A$ has full column rank
2. $A$ has a pivot in every column
3. $A$ has no free columns.
4. $\text{Null}(A) = \{0\}$
5. $Ax = 0$ has only the trivial solution.
6. $Ax = b$ has 0 or 1 soln for every $b \in \mathbb{R}^n$
7. The columns of $A$ are LI
8. $\dim \text{Col}(A) = n$
9. $\dim \text{Row}(A) = n$
Thm: TFAE:
1. $A$ has full row rank
2. $A$ has a pivot in every row
3. $A$ RER of $A$ has no zero rows
4. $\dim \text{Col}(A) = m$
5. $\text{Col}(A) = \mathbb{R}^m$
6. $Ax = b$ is consistent for every $b \in \mathbb{R}^m$
7. The columns of $A$ span $\mathbb{R}^m$
8. $\dim \text{Row}(A) = m$
9. $\text{Nul}(A^T) = \{0\}$

For a square matrix, these are the same as $m = r = n$, which means invertibility:

Thm: Let $A$ be an $n \times n$ matrix. TFAE:
1. $A$ is invertible
2. $A$ has full column rank
3. $A$ has full row rank
4. $\text{RREF}(A) = I_n$
5. There is a matrix $B$ with $AB = I_n$
6. There is a matrix $B$ with $BA = I_n$
7. $A^T$ is invertible
Eg: If $A$ is invertible then its columns
- span $\mathbb{R}^n$ (full row rank) \(\Rightarrow\) basis for $\mathbb{R}^n$
- are LI (full col rank)

Conversely, any basis for $\mathbb{R}^n$ are the columns of an invertible matrix
- spans $\Rightarrow$ full row rank
- LI $\Rightarrow$ full col rank

Basis of $\mathbb{R}^n = \text{cols of an invertible nxn matrix}$

So $\mathbb{R}^n$ has many bases! (not just $\{e_1, \ldots, e_n\}$)

NB: for an nxn matrix,
full col rank $\iff$ invertible $\iff$ full row rank

In terms of columns, $n$ vectors in $\mathbb{R}^n$
- spans $\mathbb{R}^n$ $\iff$ linearly independent

This is a special case of the basis theorem.

Basis Theorem: Let $V$ be a subspace of dim $d$
(1) If $d$ vectors span $V$ then they're a basis
(2) If $d$ vectors in $V$ are LI then they're a basis.
So if you have the correct number of vectors, you only need to check one of spans/LI.

Eg: • Two noncollinear vectors in a plane form a basis.
    • Two vectors that span a plane form a basis.