Homework #7

due Monday, October 11, at 11:59pm

1. For each column space V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\textbf{a) } V = \operatorname{Col}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad \textbf{b) } V = \operatorname{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix}$$

$$\textbf{c) } V = \operatorname{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

2. For each subspace V, compute the projection matrix P_V. Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\textbf{a) } V = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\} \qquad \textbf{b) } V = \operatorname{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

3. For each vector v, compute the projection matrix onto $V = \operatorname{Span}\{v\}$ using the formula $P_V = vv^t / v \cdot v$.

$$\textbf{a) } v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \textbf{b) } v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \qquad \textbf{c) } v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad (\text{in } \mathbf{R}^n)$$

4. **a)** Compute P_V for $V = \mathbf{R}^n$.

b) Compute P_V for $V = \{0\}$.

5. For each subspace V, compute the projection matrix P_V.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

g) The line $\{(t,-t,t) : t \in \mathbf{R}\}$.

[**Hint:** Compare Problem 9 on Homework 5 and Problem 2 on Homework 6. You can save a lot of work by sometimes computing $P_{V^\perp}$ and using $P_V = I_3 - P_{V^\perp}$.]

6. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \mathrm{Col}(A)$.

a) Compute P_V using the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \mathrm{Nul}(A^T)$.

c) Let B be the matrix with columns v_1, v_2, and compute $P_{V^\perp}$ using the formula $B(B^TB)^{-1}B^T$.

d) Verify that your answers to (a) and (c) sum to I_4.

(Factor out $ad - bc$ and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you've computed P_V, there's no need to compute $P_{V^\perp}$ separately. It's a lot of extra work!

7. Consider the plane V defined by the equation $x + 2y - z = 0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V, put your basis vectors into a matrix A, and use the formula $P_V = A(A^TA)^{-1}A^T$.

b) Compute the matrix for orthogonal projection $P_{V^\perp}$ onto the line $V^\perp$ using the formula $vv^T/v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[**Hint:** It doesn't take any work to find a basis for $V^\perp$.]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

8. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1,2,2)$ and $a_2 = (2,2,-1)$, respectively. Now compute P_1P_2, and explain why it is what it is.

9. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is symmetric and invertible, then A^{-1} is symmetric.

d) If A is any matrix, then A^TA is symmetric.

10. Decide if each statement is true or false, and explain why. In each statement, V is a subspace of $\mathbf{R}^n$.

a) The rank of P_V is equal to $\dim(V)$.

b) $P_V P_{V^\perp} = 0$.

c) $P_V + P_{V^\perp} = 0$.

d) $\mathrm{Col}(P_V) = V$.

e) $\mathrm{Nul}(P_V) = V$.

f) $\mathrm{Row}(P_V) = \mathrm{Col}(P_V)$.

g) $\mathrm{Nul}(P_V)^\perp = \mathrm{Col}(P_V)$.