Homework #5

due Wednesday, September 29, at 11:59pm

1. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

a) $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ **b)** $\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$ **c)** $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$

d) $\left\{\begin{pmatrix}1\\-2\\1\\-3\end{pmatrix}, \begin{pmatrix}2\\-4\\2\\-6\end{pmatrix}, \begin{pmatrix}3\\-5\\2\\-7\end{pmatrix}, \begin{pmatrix}-1\\4\\-3\\7\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\6\end{pmatrix}\right\}$ **e)** $\left\{\begin{pmatrix}0\\1\\3\end{pmatrix}, \begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$

Which sets do you know are linearly dependent without doing any work?

2. **a)** For each set in Problem 1, find a basis for the span of the vectors.

b) For each set in Problem 1, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for **a)**.

c) What is the dimension of each of these spans?

3. Consider the vectors

$$\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$

of Problem 1(a).

a) Find two different ways to express $(5,7,9)$ as a linear combination of these vectors.

b) How many ways can you express $(5,7,9)$ as a linear combination of the first two vectors?

4. Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V, and set

$$v_1 = w_2 + w_3 \qquad v_2 = w_1 + w_3 \qquad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V.

5. Certain vectors v_1, v_2, v_3, v_4 span a 3-dimensional subspace of $\mathbf{R}^5$. They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

a) Describe *all* linear relations among v_1, v_2, v_3, v_4.
[**Hint:** what is the rank of the matrix with columns v_1, v_2, v_3, v_4?]

b) Which vector is *not* in the span of the others, and why?

6. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ is the number of columns of A, that $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T)$ is the number of rows, and that $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

[**Hint:** Augment with the $m \times m$ identity matrix so you only have to do Gauss–Jordan elimination once.]

a) $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

7. Consider the matrix of Problem 6(b):

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

8. Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.

9. Find bases for the following subspaces.

a) $\{(x, y, x) \colon x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 \colon x = 2y + z\}$.

c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$

d) $\{x \in \mathbf{R}^3 \colon Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy-plane.

10. Let A be a 3×3 matrix of rank 2. Explain why A^2 is not the zero matrix.

[**Hint:** Compare Problem 17 on Homework 4.]

11. **a)** Let A be a 9×4 matrix of rank 3. What are the dimensions of its four fundamental subspaces?

b) If the left null space of a 5×4 matrix A has dimension 3, what is the rank of A?

12. Let V be a 4-dimensional subspace of $\mathbf{R}^5$.

a) Explain why every basis for V can be extended to a basis for $\mathbf{R}^5$ by adding one more vector.

b) Find an example of a 4-dimensional subspace V of $\mathbf{R}^5$ and a basis for $\mathbf{R}^5$ that cannot be reduced to a basis for V by removing one vector.

13. Find an example of a matrix with the required properties, or explain why no such matrix exists.

a) The column space contains $(1,2,3)$ and $(4,5,6)$, and the row space contains $(1,2)$ and $(2,3)$.

b) The column space has basis $\{(1,2,3)\}$, and the null space has basis $\{(3,2,1)\}$.

c) The dimension of the null space is one greater than the dimension of the left null space.

d) A 3×5 matrix whose row space equals its null space.

14. **a)** Show that $\text{rank}(AB) \leq \text{rank}(A)$. [**Hint:** Compare Problem 15 on Homework 4.]

b) Show that $\text{rank}(AB) \leq \text{rank}(B)$. [**Hint:** Take transposes.]

15. This problem explains why we only consider *square* matrices when we discuss invertibility.

a) Show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.

b) Show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.

[**Hint:** compare Problem 14.]

16. Decide if each statement is true or false, and explain why.

a) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension n.

b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.

c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

d) If v_3 is not a linear combination of v_1 and v_2, then $\{v_1, v_2, v_3\}$ is linearly independent.

e) If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x.

f) The set $\{0\}$ is linearly independent.

g) If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.

h) The columns of any 4×5 matrix are linearly dependent.

i) If $Ax = \binom{1}{2}$ has only one solution, then the columns of A are linearly independent.

j) If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.

k) A and A^T have the same number of pivots.

17. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full column rank"?

a) $\text{Nul}(A) = \{0\}$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \text{Row}(A) = n$

e) The columns of A span $\mathbf{R}^m$

f) A^T has full column rank

18. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full row rank"?

a) $\text{Col}(A) = \mathbf{R}^m$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \text{Nul}(A) = n - m$

e) The rows of A span $\mathbf{R}^n$

f) A^T has full column rank