Homework #5
due Wednesday, September 29, at 11:59pm

1. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.
   a) \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \]
   b) \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\} \]
   c) \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\} \]
   d) \[ \left\{ \begin{pmatrix} 1 \\ -2 \\ -3 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \\ 0 \\ 4 \\ -6 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \\ 0 \\ 3 \\ -7 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 0 \\ -1 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \]
   e) \[ \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix} \right\} \]
Which sets do you know are linearly dependent without doing any work?

2. a) For each set in Problem 1, find a basis for the span of the vectors.
   b) For each set in Problem 1, find a different basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for a).
   c) What is the dimension of each of these spans?

3. Consider the vectors
   \[ \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{pmatrix} \right\} \]
   of Problem 1(a).
   a) Find two different ways to express (5, 7, 9) as a linear combination of these vectors.
   b) How many ways can you express (5, 7, 9) as a linear combination of the first two vectors?

4. Let \( \{w_1, w_2, w_3\} \) be a basis for a subspace \( V \), and set
   \[ v_1 = w_2 + w_3 \quad v_2 = w_1 + w_3 \quad v_3 = w_1 + w_2. \]
   Show that \( \{v_1, v_2, v_3\} \) is also a basis for \( V \).

5. Certain vectors \( v_1, v_2, v_3, v_4 \) span a 3-dimensional subspace of \( \mathbb{R}^5 \). They satisfy the linear relation
   \[ 2v_1 + 0v_2 - v_3 + v_4 = 0. \]
   a) Describe all linear relations among \( v_1, v_2, v_3, v_4 \).
   [Hint: what is the rank of the matrix with columns \( v_1, v_2, v_3, v_4 \)?]
   b) Which vector is not in the span of the others, and why?
6. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that \( \dim \text{Col}(A) + \dim \text{Nul}(A) \) is the number of columns of \( A \), that \( \dim \text{Row}(A) + \dim \text{Nul}(A^T) \) is the number of rows, and that \( \dim \text{Row}(A) = \dim \text{Col}(A) \).

[Hint: Augment with the \( m \times m \) identity matrix so you only have to do Gauss-Jordan elimination once.]

\[
\begin{align*}
a) & \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} & b) & \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} & c) & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
d) & \begin{pmatrix} 1 & 2 & -3 & 1 \\ -2 & -4 & -5 & 4 \\ 1 & 2 & 2 & -3 \\ -3 & -6 & -7 & 7 \\ 6 & 1 & 12 \end{pmatrix} & e) & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}
\end{align*}
\]

7. Consider the matrix of Problem 6(b):

\[
A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}
\]

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

8. Suppose that \( A \) is an invertible \( 4 \times 4 \) matrix. Find bases for its four fundamental subspaces.

9. Find bases for the following subspaces.

a) \( \{(x, y, x): x, y \in \mathbb{R}\} \).

b) \( \{(x, y, z) \in \mathbb{R}^3: x = 2y + z\} \).

c) The solution set of the system of equations \( \begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases} \)

d) \( \{x \in \mathbb{R}^2: Ax = 2x\} \), where \( A = \begin{pmatrix} 0 & 6 & 8 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \).

e) The subspace of all vectors in \( \mathbb{R}^3 \) whose coordinates sum to zero.

f) The intersection of the plane \( x - 2y - z = 0 \) with the \( xy \)-plane.

10. Let \( A \) be a \( 3 \times 3 \) matrix of rank 2. Explain why \( A^2 \) is not the zero matrix.

[Hint: Compare Problem 17 on Homework 4.]
11. a) Let $A$ be a $9 \times 4$ matrix of rank 3. What are the dimensions of its four fundamental subspaces?

   b) If the left null space of a $5 \times 4$ matrix $A$ has dimension 3, what is the rank of $A$?

12. Let $V$ be a 4-dimensional subspace of $\mathbb{R}^5$.
   a) Explain why every basis for $V$ can be extended to a basis for $\mathbb{R}^5$ by adding one more vector.

   b) Find an example of a 4-dimensional subspace $V$ of $\mathbb{R}^5$ and a basis for $\mathbb{R}^5$ that cannot be reduced to a basis for $V$ by removing one vector.

13. Find an example of a matrix with the required properties, or explain why no such matrix exists.
   a) The column space contains $(1, 2, 3)$ and $(4, 5, 6)$, and the row space contains $(1, 2)$ and $(2, 3)$.

   b) The column space has basis $\{(1, 2, 3)\}$, and the null space has basis $\{(3, 2, 1)\}$.

   c) The dimension of the null space is one greater than the dimension of the left null space.

   d) A $3 \times 5$ matrix whose row space equals its null space.

14. a) Show that $\text{rank}(AB) \leq \text{rank}(A)$.
   [Hint: Compare Problem 15 on Homework 4.]

   b) Show that $\text{rank}(AB) \leq \text{rank}(B)$.
   [Hint: Take transposes.]

15. This problem explains why we only consider square matrices when we discuss invertibility.
   a) Show that a tall matrix $A$ (more rows than columns) does not have a right inverse, i.e., there is no matrix $B$ such that $AB = I_m$.

   b) Show that a wide matrix $A$ (more columns than rows) does not have a left inverse, i.e., there is no matrix $B$ such that $BA = I_n$.
   [Hint: compare Problem 14.]

16. Decide if each statement is true or false, and explain why.
   a) If $v_1, v_2, \ldots, v_n$ are linearly independent vectors, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ has dimension $n$.

   b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of $A$ are linearly independent.

   c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

   d) If $v_3$ is not a linear combination of $v_1$ and $v_2$, then $\{v_1, v_2, v_3\}$ is linearly independent.
e) If \( \{v_1, v_2, v_3\} \) is linearly dependent, then so is \( \{v_1, v_2, v_3, x\} \) for any vector \( x \).

f) The set \( \{0\} \) is linearly independent.

g) If \( \{v_1, v_2, v_3, v_4\} \) is linearly independent, then so is \( \{v_1, v_2, v_3\} \).

h) The columns of any \( 4 \times 5 \) matrix are linearly dependent.

i) If \( Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) has only one solution, then the columns of \( A \) are linearly independent.

j) If \( \text{Span}\{v_1, v_2, v_3\} \) has dimension 3, then \( \{v_1, v_2, v_3\} \) is linearly independent.

k) \( A \) and \( A^T \) have the same number of pivots.

17. Let \( A \) be an \( m \times n \) matrix. Which of the following are equivalent to the statement “\( A \) has full column rank”?

   a) \( \text{Nul}(A) = \{0\} \)

   b) \( A \) has rank \( m \)

   c) The columns of \( A \) are linearly independent

   d) \( \dim \text{Row}(A) = n \)

   e) The columns of \( A \) span \( \mathbb{R}^m \)

   f) \( A^T \) has full column rank

18. Let \( A \) be an \( m \times n \) matrix. Which of the following are equivalent to the statement “\( A \) has full row rank”?

   a) \( \text{Col}(A) = \mathbb{R}^n \)

   b) \( A \) has rank \( m \)

   c) The columns of \( A \) are linearly independent

   d) \( \dim \text{Nul}(A) = n - m \)

   e) The rows of \( A \) span \( \mathbb{R}^n \)

   f) \( A^T \) has full column rank