1. For each matrix $A$ and vector $b$, and express the solution set in the form
   
   \[ p + \text{Span}\{??\} \]

   for some vector $p$. For instance,

   \[
   \begin{pmatrix}
   1 & -1 \\
   2 & -2 \\
   \end{pmatrix}
   \begin{pmatrix}
   x_1 \\
   x_2 \\
   \end{pmatrix}
   =
   \begin{pmatrix}
   1 \\
   2 \\
   \end{pmatrix}
   \implies
   \begin{pmatrix}
   0 \\
   1 \\
   \end{pmatrix}
   +
   \text{Span}\left\{ \begin{pmatrix}
   1 \\
   1 \\
   \end{pmatrix} \right\}.
   \]

   [Hint: You found the parametric vector form in Problem 8 of Homework 3.]

   a) \hspace{1cm} A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \hspace{0.5cm} b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
   
   b) \hspace{1cm} A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \hspace{0.5cm} b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}
   
   c) \hspace{1cm} A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \hspace{0.5cm} b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}
   
   d) \hspace{1cm} A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \hspace{0.5cm} b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}

2. For each matrix $A$ in Problem 1, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution? Do not do Gauss–Jordan elimination again!

3. When is the following system consistent?

   \[
   \begin{align*}
   2x_1 + 2x_2 & - x_3 = b_1 \\
   -4x_1 - 5x_2 + 5x_3 & = b_2 \\
   6x_1 + x_2 + 12x_3 & = b_3
   \end{align*}
   \]

   Your answer should be a single linear equation in $b_1, b_2, b_3$. Explain the relationship between this equation and

   \[ \text{Span} \left\{ \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \right\}. \]

4. Suppose that $A$ is a $2 \times 3$ matrix such that

   \[
   A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
   \]

   a) Find two different solutions of $Ax = 0$. 
b) Find two more solutions of \( Ax = \begin{pmatrix} -1 \end{pmatrix} \).

5. Suppose that \( Ax = b \) is consistent. Explain why \( Ax = b \) has a unique solution precisely when \( Ax = 0 \) has only the trivial solution.

6. Give geometric descriptions of the following spans (line, plane, ...).

   a) Span \( \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \)
   b) Span \( \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \)
   c) Span \( \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} \)
   d) Span \( \begin{pmatrix} -4 \\ -5 \\ 6 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix} \)
   e) Span \( \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \)

   [Hint: for d), compare Problem 3.]

7. a) List five nonzero vectors contained in \( \text{Span} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \). 
   
   b) Is \( \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \) contained in \( \text{Span} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \)?

      If so, express \( \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix} \) as a linear combination of \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \).

   c) Show that \( \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \) is contained in \( \text{Span} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \).

   d) Describe \( \text{Span} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \) geometrically.

   e) Find a vector not contained in \( \text{Span} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \).

8. Decide if each statement is true or false, and explain why.

   a) A vector \( b \) is a linear combination of the columns of \( A \) if and only if \( Ax = b \) has a solution.

   b) There is a matrix \( A \) such that \( Ax = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) has infinitely many solutions and \( Ax = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \) has exactly one solution.

   c) The zero vector is contained in every span.
d) The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what $A$ is.

e) If the zero vector is a solution of a system of equations, then the system is homogeneous.

f) If $Ax = b$ has a unique solution, then $A$ has a pivot in every column.

g) If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

h) It is possible for $Ax = b$ to have exactly 13 solutions.

9. Find a spanning set for the null space of each matrix, and express the null space as the column space of some other matrix.

$$
\begin{align*}
\text{a)} & \quad \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \\
\text{b)} & \quad \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \\
\text{c)} & \quad \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \\
\text{d)} & \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}
\end{align*}
$$

[Hint: Compare Problem 2.]
10. Draw pictures of the null space and the column space of the following matrices. Be precise!

\[ A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \]

\[ Nul(A) \quad Col(A) \]

\[ A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} : \]

\[ Nul(A) \quad Col(A) \]

\[ A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \]

\[ Nul(A) \quad Col(A) \]

11. Give examples of subsets \( V \) of \( \mathbb{R}^2 \) such that:
   a) \( V \) is closed under addition and contains 0, but is not closed under scalar multiplication.
   b) \( V \) is closed under scalar multiplication and contains 0, but is not closed under addition.
   c) \( V \) is closed under addition and scalar multiplication, but does not contain 0. Therefore, none of these conditions is redundant.

12. Which of the following subsets of \( \mathbb{R}^3 \) are subspaces? If it is not a subspace, why not? If it is, write it as the column space or null space of some matrix.
   a) The plane \( \{(x, y, x) : x, y \in \mathbb{R}\} \).
   b) The plane \( \{(x, y, 1) : x, y \in \mathbb{R}\} \).
   c) The set consisting of all vectors \((x, y, z)\) such that \(xy = 0\).
   d) The set consisting of all vectors \((x, y, z)\) such that \(x \leq y\).
e) The span of (1, 2, 3) and (2, 1, −3).

f) The solution set of the system of equations \[
\begin{align*}
x + y + z &= 0 \\
x - 2y - z &= 0.
\end{align*}
\]

g) The solution set of the system of equations \[
\begin{align*}
x + y + z &= 0 \\
x - 2y - z &= 1.
\end{align*}
\]

13. Give a geometric description of the following column spaces (line, plane, …).

a) \( \text{Col} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \)

b) \( \text{Col} \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} \)

c) \( \text{Col} \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & -6 \end{pmatrix} \)

d) \( \text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \)

e) \( \text{Col} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \)

[Hint: Compare Problem 6.]

14. Find a nonzero \( 2 \times 2 \) matrix such that \( A^2 = 0 \).

15. a) Explain why \( \text{Col}(AB) \) is contained in \( \text{Col}(A) \).

b) Give an example where \( \text{Col}(AB) \neq \text{Col}(A) \). Can you take \( A = B \)?

[Hint: use Problem 14.]

16. a) Explain why \( \text{Nul}(AB) \) contains \( \text{Nul}(B) \).

b) Give an example where \( \text{Nul}(AB) \neq \text{Nul}(B) \). Can you take \( A = B \)?

[Hint: use Problem 14.]

17. a) If \( \text{Col}(B) \) is contained in \( \text{Nul}(A) \), then \( AB = \) ________.

b) Find a \( 2 \times 2 \) matrix \( A \) such that \( \text{Col}(A) = \text{Nul}(A) \). What is the rank of such a matrix? [Hint: use Problem 14.]

18. Find a matrix \( A \) such that

\[
\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.
\]

What is the rank of \( A \)?

19. For the following matrix \( A \), compute the reduced row echelon form of \( A \) and of \( A^T \). Do they have the same free variables? Do they have the same rank?

\[
A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}
\]
20. Decide if each statement is true or false, and explain why.
   a) The null space of an $m \times n$ matrix with $n$ pivots is $\mathbb{R}^n$.
   b) If $\text{Col}(A) = \{0\}$, then $A$ is the zero matrix.
   c) The column space of $2A$ equals the column space of $A$.
   d) The null space of $A + B$ contains the null space of $A$.
   e) If $U$ is an echelon form of $A$, then $\text{Nul}(U) = \text{Nul}(A)$.
   f) If $U$ is an echelon form of $A$, then $\text{Col}(U) = \text{Col}(A)$. 