Homework #3

due Monday, September 13, at 11:59pm

1. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax.

a)

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

2. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

3. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)

$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

4. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing maximal partial pivoting. Check your answers by multiplying PA and LU.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

5. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.

b) If P is any permutation matrix, show that $P^{-1} = P^T$. [**Hint:** write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i.] Is $P = P^T$ for a general permutation matrix?

c) Explain why a permutation matrix has exactly one 1 in each row and in each column, with all other entries equal to zero. Is any matrix of this form a permutation matrix? Why or why not?

d) Let A be an $m \times n$ matrix and let P be the $n \times n$ elementary matrix for swapping row i and row j. Show that AP is the matrix obtained from A by swapping *columns* i and j. [**Hint:** $(AP)^T = PA^T$.]

6. Consider the matrix

$$A = \begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix}.$$

In this problem, we will see how to produce a $PA = LU$ decomposition of A using elementary matrices.

a) Perform Gaussian elimination on A using maximal partial pivoting. Write down the elementary matrices you use. You should end up with

$$U = \begin{pmatrix} 3 & 0 & -14 & -8 \\ 0 & 8 & -17 & 28 \\ 0 & 0 & -\frac{19}{12} & \frac{26}{3} \\ 0 & 0 & 0 & \frac{3}{19} \end{pmatrix} = E_4 P_3 E_3 P_2 E_2 E_1 P_1 A$$

where P_1, P_2, P_3 are elementary matrices for row swaps and E_1, E_2, E_3, E_4 correspond to row replacements.

b) Explain why $P_2 E_2 E_1 = (P_2 E_2 E_1 P_2) P_2$. Compute $P_2 E_2 E_1 P_2$, and verify that it is lower-unitriangular. (Do not multiply matrices: perform row operations!)

c) Compute $P_3 E_3 (P_2 E_2 E_1 P_2) P_3$ using (b) and using Problem 5(d) to multiply on the left and right by permutation matrices. Verify that it is lower-unitriangular.

d) Rewrite $U = E_4 P_3 E_3 P_2 E_2 E_1 P_1 A$ as

$$U = E_4\big(P_3 E_3 (P_2 E_2 E_1 P_2) P_3\big) P_3 P_2 P_1 A$$

Which products of elementary matrices are L^{-1} and P?

7. For each matrix A and vector b, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

d) $$A=\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b=\begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

e) $$A=\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b=\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

8. For each matrix A and vector b in Problem 7, find the parametric *vector* form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the dimension of the solution set?

9. The equation $x + 2y = z$ determines a plane in $\mathbf{R}^3$. (This is an *implicit equation* for the plane).

a) What is the coefficient matrix A for this system?

b) Which are the free variables?

c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.

d) Do the same for the plane defined by $2y = z$. What is different?

10. The equations

$$\begin{aligned} x + \ \ y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line $\mathbf{R}^3$. (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t. (Use the free variable as the parameter t.)

11. Express each system of linear equations as a vector equation. For example,

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 - \ \ x_2 &= 4 \end{aligned} \rightsquigarrow x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

a) $$\begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 \qquad\quad + 4x_3 = 2 \end{cases}$$

b) $$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

c) $$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$$

12. **a)** Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?
If so, what are the coefficients?

b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 7.]

13. Describe and compare (geometrically) the solution sets of the following systems:

$$\begin{cases} 2x_1 + x_2 + x_3 = 1 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases} \qquad \begin{cases} 6x_1 + 3x_2 + 2x_3 = 2 \\ 2x_1 + x_2 = 1 \end{cases}$$

This is much easier if you write the solutions in parametric vector form.

14. Let A be a 4×5 matrix with four pivots. Suppose that

$$(\text{column } 1) + 2(\text{column } 3) - (\text{column } 4) = 0.$$

a) Find a nonzero solution of $Ax = 0$.

b) Which is the free variable?
[**Hint:** the solutions of $Ax = 0$ are unchanged by Gauss–Jordan elimination.]

15. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \qquad t \in \mathbf{R}.$$

16. Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} x = b$$

is consistent.

17. Suppose that A is a 3×3 matrix and b is a vector such that $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

18. Give examples of matrices A in reduced row echelon form for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

19. Decide if each statement is true or false, and explain why.

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An $m \times n$ matrix has at most m pivots.

d) A wide matrix (more columns than rows) must have a free variable.

e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.