Homework #14

due Wednesday, December 1, at 11:59pm

1. For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

[**Hint:** one of the singular values in **e)** is 12.]

2. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 1(a). Let σ_1, σ_2 be the singular values of A.

a) Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

b) Find an orthonormal eigenbasis $\{v_1, v_2\}$ of $A^T A$ such that $A^T A v_i = \sigma_i^2 v_i$ and an orthonormal eigenbasis $\{u_1, u_2\}$ of AA^T such that $AA^T u_i = \sigma_i^2 u_i$, but where A is *not* equal to $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

[**Hint:** The condition $Av_i = \sigma_i u_i$ is not automatic!]

3. Find the matrix A satisfying

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

and write the SVD of A in outer product form.

[**Hint:** Start by finding the SVD.]

4. Let A be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.

5. **a)** Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$. What is $\det(A^T A)$?]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in Problem 1 works.]

6. Let S be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i.

- **a)** Show that the singular values of S are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\text{rank}(S) = r$.
- **b)** Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i.

7.

- **a)** Show that all singular values of an orthogonal matrix are equal to 1.
- **b)** Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use Problem 11 on Homework 10.]

  Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

8. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

- **a)** Show that A and R have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.
- **b)** What is the relationship between the left singular vectors of A and R?

9. Let A be a matrix with first singular value σ_1 and first right singular vector v_1.

- **a)** Show that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is the same as the maximum value of $\|Ax\|/\|x\|$ subject to $x \neq 0$.
- **b)** Show that $\|Ax\|/\|x\|$ is maximized at $x = v_1$, with maximum value σ_1.
  [**Hint:** How do you maximize $\|Ax\|^2 = x^T(A^T A)x$ for $\|x\| = 1$?]
- **c)** Suppose now that A is square and λ is an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

  This shows that *the largest singular value is at least as big as the largest eigenvalue.*

Remark: The maximum value of $\|Ax\|/\|x\|$ for $x \neq 0$ is called the *norm* of A and is denoted $\|A\|$.

10.

- **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are not*. This is another advantage of the SVD.

11. Decide if each statement is true or false, and explain why.

a) The left singular vectors of A are eigenvectors of $A^T A$ and the right singular vectors are eigenvectors of AA^T.

b) For any matrix A, the matrices AA^T and $A^T A$ have the same nonzero eigenvalues.

c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.

d) If A does not have full column rank, then 0 is a singular value of A.

e) Suppose that A is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of A are orthogonal to Nul(A).