Homework #14
Answer Key

1. For each matrix $A$, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$ 

a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$  
b) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$  
c) $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$  
d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$  
e) $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

[Hint: one of the singular values in e) is 12.]

Solution.

a) $
\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} = 10\sqrt{2} \begin{pmatrix} 1 \\ \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 5\sqrt{2} \begin{pmatrix} 1 \\ \sqrt{5} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{10} \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} 
$

b) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} = 5\sqrt{2} \begin{pmatrix} 1 \\ \sqrt{5} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{10} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} 
$

c) $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix} = 3\sqrt{26} \begin{pmatrix} 1 \\ 3\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{13} \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} 
+ \sqrt{78} \begin{pmatrix} 1 \\ \sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{13} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} 
$

d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} = 5\sqrt{15} \begin{pmatrix} 4 \\ -3 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{15} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} 
+ 5\sqrt{6} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{6} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \\ 0 \end{pmatrix} 
$

e) $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix} = 12 \begin{pmatrix} 2 \\ \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} 
+ 6 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} 
+ 6 \begin{pmatrix} 1 \\ \sqrt{18} \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} 
$

2. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$
of Problem 1(a). Let $\sigma_1, \sigma_2$ be the singular values of $A$.

**a)** Find all singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

**b)** Find an orthonormal eigenbasis $\{v_1, v_2\}$ of $A^T A$ such that $A^T A v_i = \sigma_i^2 v_i$ and an orthonormal eigenbasis $\{u_1, u_2\}$ of $A A^T$ such that $A A^T u_i = \sigma_i^2 u_i$, but where $A$ is not equal to $\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

**[Hint: The condition $A v_i = \sigma_i u_i$ is not automatic!]**

**Solution.**

**a)** The only ambiguity in the SVD is the choice of right singular vectors $v_1, v_2$. There are two choices for each, and these determine $u_1 = \sigma_1^{-1} A v_1$ and $u_2 = \sigma_2^{-1} A v_2$. Hence the general SVD is

$$\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} = 10\sqrt{2} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + 5\sqrt{2} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

**b)** We just have to choose the wrong $u_1, u_2$ given a choice of $v_1, v_2$:

$$\begin{pmatrix} -8 \\ -1 \\ -11 \end{pmatrix} = 10\sqrt{2} \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + 5\sqrt{2} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

3. Find the matrix $A$ satisfying

$$A \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix},$$

and write the SVD of $A$ in outer product form.

**[Hint: Start by finding the SVD.]**

**Solution.**

Note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We take $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. We want $u_1 = \sigma_1 A v_1$ and $u_2 = \sigma_2 A v_2$, and we have

$$A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{3}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad A v_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{3}{2\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This means $\sigma_1 = 3/\sqrt{2}$, $v_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\sigma_2 = 3/2\sqrt{2}$, and $v_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so the SVD is:

$$A = \frac{3}{\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2\sqrt{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 4 \\ 0 \\ 5 \\ 3 \\ 2 \\ 6 \end{pmatrix}.$$

4. Let $A$ be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of $A$ in outer product form.

**Solution.**
Let \( q_i = w_i / \sigma_i \), the unit vector in the direction of \( w_i \). Then
\[
A = \sigma_1 q_1 e_1^T + \sigma_2 q_2 e_2^T + \cdots + \sigma_n q_n e_n^T.
\]

5. a) Let \( A \) be an invertible \( n \times n \) matrix. Show that the product of the singular values of \( A \) equals the absolute value of the product of the (real and complex) eigenvalues of \( A \) (counted with algebraic multiplicity).

[Hint: Both equal \( |\det(A)| \). What is \( \det(A^T A) \)?]

b) Find an example of a \( 2 \times 2 \) matrix \( A \) with distinct positive eigenvalues that are not equal to any of the singular values of \( A \).

[Hint: One of the matrices in Problem 1 works.]

Solution.

a) Let \( A = U\Sigma V^T \) be the SVD in matrix form. Note that \( A \) has \( n \) singular values because it has full rank. We have \( \det(U) = \pm 1 \) and \( \det(V) = \pm 1 \), so
\[
\det(A) = \pm \det(\Sigma) = \pm \sigma_1 \cdots \sigma_n.
\]

Hence the product of the singular values of \( A \) equals \( |\det(A)| \). The absolute value of the product of the eigenvalues of \( A \) (counted with multiplicity) also equals \( |\det(A)| \) by HW11#12(a).

b) Taking \( A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \), the matrix of Problem 1(a), the singular values are \( 10 \sqrt{2} \approx 14.142 \) and \( 5 \sqrt{2} \approx 7.071 \), and the eigenvalues are \( \frac{1}{2}(21 + \sqrt{41}) \approx 13.702 \) and \( \frac{1}{2}(21 - \sqrt{41}) \approx 7.298 \).

6. Let \( S \) be a symmetric matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) (counted with multiplicity). Order the eigenvalues so that \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n \). Let \( \{v_1, \ldots, v_n\} \) be an orthonormal eigenbasis, where \( v_i \) has eigenvalue \( \lambda_i \).

a) Show that the singular values of \( S \) are \( |\lambda_1|, \ldots, |\lambda_r| \). In particular, \( \text{rank}(S) = r \).

b) Find the singular value decomposition of \( S \) in outer product form, in terms of the \( \lambda_i \) and the \( v_i \).

Solution.

a) The nonzero eigenvalues of \( S^T S = S^2 \) are \( \lambda_1^2, \ldots, \lambda_r^2 \), so the singular values are \( \sigma_1 = \sqrt{\lambda_1^2} = |\lambda_1|, \ldots, \sigma_r = \sqrt{\lambda_r^2} = |\lambda_r| \).

b) The right singular vectors of \( S \) are \( v_1, \ldots, v_r \). We Have
\[
\frac{1}{\sigma_i} A v_i = \frac{1}{|\lambda_i|} \lambda_i v_i = \pm v_i,
\]
with the positive sign when \( \lambda_i > 0 \) and a negative sign otherwise. Hence the SVD is
\[
S = \sum_{i=1}^r |\lambda_i| \frac{\lambda_i}{|\lambda_i|} v_i v_i^T = \sum_{i=1}^r |\lambda_i| (\pm v_i) v_i^T.
\]
7. **a)** Show that all singular values of an orthogonal matrix are equal to 1.

**b)** Let $A$ be an $m \times n$ matrix, let $Q_1$ be an $m \times m$ orthogonal matrix, and let $Q_2$ be an $n \times n$ orthogonal matrix. Show that $A$ has the same singular values as $Q_1AQ_2$.

**[Hint:** Use HW10#11.]

**Remark:** This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

**Solution.**

**a)** Suppose that $Q$ is orthogonal. Then $Q^TQ = I_n$, whose only eigenvalue is 1. It follows that 1 is the only singular value of $Q$.

**b)** The matrix $(Q_1AQ_2)^T(Q_1AQ_2) = Q_2^TA^TQ_1^TQ_1AQ_2 = Q_2^TA^TAQ_2$ has the same eigenvalues as $A^TA$ since $Q_2^T = Q_2^{-1}$: see HW10#11. Hence $Q_1AQ_2$ has the same singular values as $A$.

8. Let $A$ be a matrix of full column rank and let $A = QR$ be the QR decomposition of $A$.

**a)** Show that $A$ and $R$ have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

**b)** What is the relationship between the left singular vectors of $A$ and $R$?

**Solution.**

**a)** We have $A^TA = (QR)^T(QR) = R^TQ^TQR = R^TR$. In particular, $A^TA$ and $R^TR$ have the same eigenvalues and eigenvectors, so $A$ and $R$ have the same singular values and right singular vectors.

**b)** Let $v$ be a right singular vector of $R$ (and $A$) with singular value $\sigma$. The associated left singular vector of $R$ is $u = \frac{1}{\sigma}Rv$, and of $A$ is $\frac{1}{\sigma}Av = \frac{1}{\sigma}QRv = Qu$. Hence the left singular vectors of $A$ are obtained from the left singular vectors of $R$ by multiplication by $Q$.

9. Let $A$ be a matrix with first singular value $\sigma_1$ and first right singular vector $v_1$.

**a)** Show that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is the same as the maximum value of $\|Ax\|/\|x\|$ subject to $x \neq 0$.

**b)** Show that $\|Ax\|/\|x\|$ is maximized at $x = v_1$, with maximum value $\sigma_1$.

**[Hint:** How do you maximize $\|Ax\|^2 = x^T(A^TA)x$ for $\|x\| = 1$?]

**c)** Suppose now that $A$ is square and $\lambda$ is an eigenvalue of $A$. Show that $|\lambda| \leq \sigma_1$.

(You may assume $\lambda$ is real, although it is also true for complex eigenvalues.)

This shows that the largest singular value is at least as big as the largest eigenvalue.

**Remark:** The maximum value of $\|Ax\|/\|x\|$ for $x \neq 0$ is called the *norm* of $A$ and is denoted $\|A\|$.
Solution.

a) If $x \neq 0$ then
\[
\frac{\|Ax\|}{\|x\|} = \frac{\|A\frac{x}{\|x\|}\|}{\|x\|},
\]
so $\|Ax\|/\|x\|$ is equal to $\|Ay\|$ for a unit vector $y$.

b) Maximizing $\|Ax\|$ subject to $\|x\| = 1$ is the same as maximizing
\[
\|Ax\|^2 = (Ax) \cdot (Ax) = x^T A^T A x
\]
subject to $\|x\| = 1$. This is a quadratic optimization problem with matrix $S = A^T A$; the maximum value is obtained at a unit eigenvector with largest eigenvalue. This is the same as a right singular vector of $A$ with largest singular value. The maximum value is then $\|Av_1\| = \sigma_1$.

c) Suppose $Ax = \lambda x$ for $x \neq 0$. By b),
\[
\sigma_1 \geq \frac{\|Ax\|}{\|x\|} = \frac{\|\lambda\|\|x\|}{\|x\|} = |\lambda|.
\]

10. a) Find the eigenvalues and singular values of
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

b) Find the (real and complex) eigenvalues and singular values of
\[
A' = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0.0001 & 0 & 0 & 0
\end{pmatrix}.
\]

c) Note that $A$ is very close to $A'$ numerically. Were the eigenvalues of $A$ close to the eigenvalues of $A'$? What about the singular values?

This problem is meant to illustrate the fact that eigenvalues are numerically unstable but singular values are not. This is another advantage of the SVD.

Solution.

a) This matrix is upper-triangular with zero entries on the diagonal; its eigenvalues are therefore all zero. We have
\[
A^T A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
so there is one singular value $\sigma = 1$. 
b) The characteristic polynomial of $A'$ is $\lambda^4 - 0.0001$, which has roots $\pm 0.1$ and $\pm 0.1i$. We have

$$A'^TA' = \begin{pmatrix} 0.0001 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so the singular values are 1 and 0.0001.

c) The eigenvalues of $A'$ were within 0.1 of the eigenvalues of $A$, whereas the singular values were within 0.0001: the eigenvalues changed by three orders of magnitude more than the singular values (which only changed as much as the matrix did).

11. Decide if each statement is true or false, and explain why.

a) The left singular vectors of $A$ are eigenvectors of $A^TA$ and the right singular vectors are eigenvectors of $AA^T$.

b) For any matrix $A$, the matrices $AA^T$ and $A^TA$ have the same nonzero eigenvalues.

c) If $S$ is symmetric, then the nonzero eigenvalues of $S$ are its singular values.

d) If $A$ does not have full column rank, then 0 is a singular value of $A$.

e) Suppose that $A$ is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of $A$ are orthogonal to $\text{Nul}(A)$.

Solution.

a) False: it’s the other way around.

b) True: they are the squares of the singular values of $A$.

c) False: you have to take the absolute value.

d) False: zero is not considered a singular value.

e) False: for instance, the singular values of $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ are $10\sqrt{2} \approx 14.142$ and $5\sqrt{2} \approx 7.071$ but the singular values of $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} + I_2$ are $\approx 15.133$ and $\approx 8.062$.

f) True: they form a basis for $\text{Row}(A) = \text{Nul}(A)^\perp$. 