Homework #13

due Wednesday, November 24, at 11:59pm

1. For each symmetric matrix S, decide if S is positive-definite. If so, find its LDL^T and Cholesky decompositions. Do not compute any eigenvalues!

$$\textbf{a)}\ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} -1 & 2 & 3 & -2 \\ 2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$$

2. Consider the matrix

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Without multiplying the matrices, find:

a) The determinant of S.

b) The eigenvalues of S.

c) The eigenvectors of S.

d) A reason why S is symmetric positive-definite.

3. **a)** For each symmetric matrix S, compute the associated quadratic form $q(x) = x^T Sx$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

b) Let A be a square matrix and let $S = \frac{1}{2}(A + A^T)$. Show that S is symmetric and that $x^T Ax = x^T Sx$. (This is why we only consider symmetric matrices when studying quadratic forms.)

4. For each quadratic form $q(x_1, x_2)$, **i)** write $q(x)$ in the form $x^T Sx$ for a symmetric matrix S, **ii)** find coordinates y_1, y_2 such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, **iii)** draw the solutions of $q(x_1, x_2) = 1$, being sure to draw the shortest and longest solutions, and **iv)** find the maximum and maximum values of $q(x_1, x_2)$ subject to the constraint $x_1^2 + x_2^2 = 1$, and at which points (x_1, x_2) these values are attained.

a) $q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2$

b) $q(x_1, x_2) = \dfrac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2)$

c) $q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2$

[**Hint:** An equation of the form $(x_1/r_1)^2 - (x_2/r_2)^2 = 1$ defines a hyperbola.]

5. For the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3,$$

find coordinates y_1, y_2, y_3 such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$, and find the maximum and minimum values of $x_1^2+x_2^2+x_3^2$ subject to the constraint $q(x_1, x_2, x_3) = 1$, along with the points (x_1, x_2, x_3) at which these values are attained.

6. Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3.$$

Find all vectors $x = (x_1, x_2, x_3)$ maximizing $q(x)$ subject to $\|x\| = 1$. (There are infinitely many!)

7. In this problem, we will touch on the role of quadratic optimization in *spectral graph theory*. Spectral graph theory is the study of graphs using linear algebra, and is widely applied to problems in networking and partitioning. (Google's PageRank algorithm is one such application.)

A *graph* is a set of *vertices*, or points, connected by a set of *edges*. For simplicity, we will assume that each edge has distinct endpoints (i.e., there are no loop edges), and that there is at most one edge connecting any two vertices: such a graph is called *simple*. Under these assumptions, an edge is determined by the two vertices it connects, so we can write $e = (1, 2)$ for the edge connecting vertices 1 and 2. We also write $i \sim j$ if (i, j) is an edge of G. The *degree* of a vertex is the number of edges connected to it; the degree of vertex i is written $\deg(i)$.

Let G be a graph with n vertices labeled $1, 2, \ldots, n$. We consider a vector $x \in \mathbf{R}^n$ as a way to assign a real number to each vertex: the ith coordinate x_i is the number attached to the ith vertex. The *Laplacian* of G is the $n \times n$ matrix L whose (i, j) entry is

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that L is symmetric. Let $x \in \mathbf{R}^n$ and let $y = Lx$. Then the ith coordinate of y is

$$(\star) \qquad y_i = x_i \deg(i) - \sum_{j \sim i} x_j = \sum_{j \sim i} (x_i - x_j).$$

In other words, y is the vector that assigns the number $\sum_{j \sim i}(x_i - x_j)$ to vertex i.

The eigenvalues of the graph Laplacian contain important information about the structure of the graph.

a) Show that the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^n$ is in the null space of L.

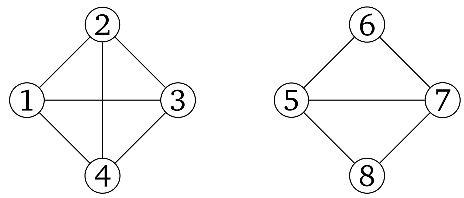
It follows that 0 is always an eigenvalue of L. More interesting is the *second-smallest* eigenvalue.

b) Show that $x^T L x = \sum_{j \sim i} (x_i - x_j)^2$. Explain why L is positive-semidefinite.

This means that minimizing $q(x) = x^T L x$, subject to the constraints $\|x\|^2 = 1$ and $x \perp \mathbf{1}$, amounts to finding a way to assign a number to each vertex such that *neighboring vertices have similar values*, but such that the sum of the values is zero and the sum of their squares is 1.

For each of the following graphs, **i)** compute the Laplacian matrix L and **ii)** minimize $x^T L x$ subject to $x \perp \mathbf{1}$ and $\|x\| = 1$. **iii)** For a (unit) vector x achieving this minimum, draw the number x_i next to vertex i on the graph. **iv)** What does the second-smallest eigenvalue say about the graph? (This is open-ended.)

c)



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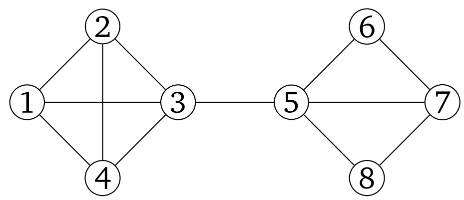

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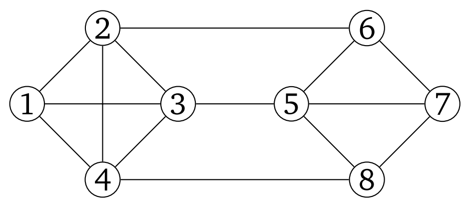

You should feel free to use a computer algebra system to compute the eigenvalues and eigenvectors. For instance, you can use SymPy in the Sage cell on the course webpage. Finding the eigenvalues and eigenvectors of a matrix in SymPy is done as follows: if your matrix is

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

then you would type:

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A = Matrix([[7,2,0],[2,6,2],[0,2,5]]) * 1.0
pprint(A.eigenvects())
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(The " * 1.0" forces Sympy to output a decimal answer instead of a symbolic one.) The output is a list of tuples of the form (eigenvalue, multiplicity, eigenspace basis)—note that the eigenspace basis will not necessarily be orthonormal.